Commutants of Finite Blaschke Product Multiplication Operators on Hilbert Spaces of Analytic Functions

Carl C. Cowen

IUPUI

(Indiana University Purdue University Indianapolis)

ICM Satellite Conference on Operator Algebras and Applications

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joint work with Rebecca Wahl (Butler University)

In this talk $\mathcal{H}$ will denote a Hilbert space of analytic functions on $\mathbb{D}$,

Usual spaces: f analytic in $\mathbb{D}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$\text{Hardy:}\quad H^2(\mathbb{D}) = H^2 = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

$$\text{Bergman:}\quad A^2(\mathbb{D}) = A^2 = \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Bergman } (\gamma > -1):\quad A^2_\gamma = \{f : \|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\gamma \frac{dA(z)}{\pi} < \infty\}$$

$$\text{weighted Hardy } (\|z^n\| = \omega_n > 0):\quad H^2(\omega) = \{f : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \omega_n^2 < \infty\}$$

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In this talk, we will consider spaces H^2_κ for $\kappa \geq 1$ which are the weighted Hardy spaces with

$$K_\alpha(z) = (1 - \overline{\alpha} z)^{-\kappa}$$

The spaces H^2_κ include the usual Hardy and Bergman spaces and all the weighted Bergman spaces ($\gamma = \kappa + 2$).

Conversation with Axler made it clear that right generality is to consider Hilbert spaces, $\mathcal{H}$, of functions analytic on $\mathbb{D}$ that satisfy:

(I) The constant function $1(z) \equiv 1$ for z in $\mathbb{D}$ is in $\mathcal{H}$ and $\|1\| = 1$

(II) For α in $\mathbb{D}$, the linear functional $f \mapsto f(\alpha)$ is continuous on $\mathcal{H}$

(III) For ψ in H^∞, operator T_ψ given by $(T_\psi f)(z) = \psi(z)f(z)$ is in $\mathcal{B}(\mathcal{H})$.

(IV) For α in $\mathbb{D}$ and f in $\mathcal{H}$ with $f(\alpha) = 0$, then $f/(z-\alpha)$ is also in $\mathcal{H}$.

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- Conditions (I) & (III) say $\mathcal{H}$ and its multiplier algebra contain H^∞
- Condition (II) says $\mathcal{H}$ has kernel functions & its multiplier algebra is H^∞
- For ψ in H^∞, the operator T_ψ in condition (III) is called an *analytic multiplication operator* or an *analytic Toeplitz operator* and conditions imply $\|T_\psi\| = \|\psi\|_\infty$ and this means $\|\psi\| \leq \|\psi\|_\infty$

The Hardy space H^2, the Bergman space A^2, and the standard weight Bergman spaces H^2_κ satisfy Conditions (I), (II), (III), and (IV).

The usual Dirichlet space, and many weighted Dirichlet spaces, do not satisfy all the conditions: not all H^∞ functions are in Dirichlet space!

The Hardy space H^2, the Bergman space A^2, and the standard weight Bergman spaces H^2_κ satisfy Conditions (I), (II), (III), and (IV).

Consequence: if f is in $\mathcal{H}$, ψ is bounded analytic function, and α is in $\mathbb{D}$,

$$\langle f, T_\psi^* K_\alpha\rangle = \langle T_\psi f, K_\alpha\rangle = \psi(\alpha) f(\alpha) = \psi(\alpha)\langle f, K_\alpha\rangle = \langle f, \overline{\psi(\alpha)} K_\alpha\rangle$$

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Since f is arbitrary, this means $\quad T_\psi^* K_\alpha = \overline{\psi(\alpha)} K_\alpha$

and every kernel function is an eigenvector for T_ψ^*.

The spectrum of T_ψ is the closure of $\psi(\mathbb{D})$, there no eigenvalues for T_ψ, but the complex conjugate of $\psi(\mathbb{D})$ consists of eigenvalues of T_ψ^*.

Definition:

An *inner function* is a bounded analytic function, ψ, on $\mathbb{D}$ such that

$$\lim_{r \to 1^-} |\psi(re^{i\theta})| = 1 \quad \text{a. e.} \quad d\theta$$

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Definition:

A function B is a *Blaschke product of order* n if it can be written as

$$B(z) = \mu \left(\frac{\zeta_1 - z}{1 - \overline{\zeta_1} z} \right) \left(\frac{\zeta_2 - z}{1 - \overline{\zeta_2} z} \right) \cdots \left(\frac{\zeta_n - z}{1 - \overline{\zeta_n} z} \right)$$

where $|\mu| = 1$ and $\zeta_1, \zeta_2, \cdots, \zeta_n$ are points of $\mathbb{D}$.

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Blaschke products of order n are inner functions

and map the closed disk n-to-1 onto itself.

For ψ, a non-constant inner function, the multiplication operator T_ψ is a pure isometry on H^2 but is *not* isometric on the Bergman spaces.

Beurling's Theorem (1949):

Let T_z be the operator of multiplication by z on $H^2(\mathbb{D})$. A closed subspace M of $H^2(\mathbb{D})$ is invariant for T_z if and only if there is an inner function ψ such that $M = \psi H^2(\mathbb{D})$.

This result is indicative of the interest in the operator T_z of multiplication by z on $H^2(\mathbb{D})$ and in analytic Toeplitz operators T_ψ on Hilbert spaces of analytic functions more generally.

Definition:

If A is a bounded operator on a space $\mathcal{H}$, the *commutant of* A is the set

$\{A\}' = \{S \in \mathcal{B}(\mathcal{H}) : AS = SA\}$

For example, for T_z on H^2,

$$\{T_z\}' = \{T_\psi : \psi \in H^\infty\}$$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \ddots \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \ddots \end{pmatrix}$$

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This means that $a_{0j} = 0$ for $j \geq 1$ and $a_{i,j} = a_{i+1,j+1}$ for $i, j \geq 0$

In particular, the matrix is lower triangular and is constant along diagonals:

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

This is T_ψ for $\psi(z) = \sum_{j=0}^{\infty} a_j z^j$ where $\|\psi\|_\infty = \|T_\psi\|$.

Definition:

If A is a bounded operator on a space $\mathcal{H}$, the *commutant of* A is the set

$\{A\}' = \{S \in \mathcal{B}(\mathcal{H}) : AS = SA\}$

We have seen for T_z on H^2,

$$\{T_z\}' = \{T_\psi : \psi \in H^\infty\}$$

By the 1970's, there was interest in the more general question,

For ψ in H^∞ and T_ψ an operator on H^2, what is $\{T_\psi\}'$?

or more specifically,

For B a finite Blaschke product and T_B operating on H^2, what is $\{T_B\}'$?

Deddens & Wong's 1973 paper used the fact that for B a finite Blaschke product, the operator T_B acting on H^2 is a pure isometry to show that

The operator S in $\mathcal{B}(H^2)$ is in $\{T_B\}'$ if and only if

S can be represented as a lower triangular block Toeplitz matrix with respect to the description of H^2 as $\bigoplus_{k=0}^{\infty} B^k\mathcal{W}$ where $\mathcal{W}$ is the wandering subspace $\mathcal{W} = (BH^2)^{\perp}$, that is,

$$S = \begin{pmatrix} A_0 & 0 & 0 & 0 & \cdots \\ A_1 & A_0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & \cdots \\ A_3 & A_2 & A_1 & A_0 & \cdots \\ & \vdots & & & \ddots \end{pmatrix}$$

Shortly thereafter, Thomson's papers and Cowen's papers computed $\{T_B\}'$ from a different perspective:

Fundamental Lemma:

For S a bounded operator on H^2 and ψ in H^∞, these • are equivalent

- *S commutes with T_ψ*
- *For all α in $\mathbb{D}$, $S^* K_\alpha \perp (\psi - \psi(\alpha))H^2$*

Proof: (Main calculation)

For α in $\mathbb{D}$, ψ in H^∞, and $ST_\psi = T_\psi S$, if f is in H^2,

$$\langle(\psi - \psi(\alpha))f, S^* K_\alpha\rangle = \langle ST_\psi f, K_\alpha\rangle - \psi(\alpha)\langle Sf, K_\alpha\rangle$$

$$= \langle T_\psi Sf, K_\alpha\rangle - \psi(\alpha)\langle Sf, K_\alpha\rangle = \langle Sf, T_\psi^* K_\alpha\rangle - \psi(\alpha)\langle Sf, K_\alpha\rangle$$

$$= \psi(\alpha)(Sf)(\alpha) - \psi(\alpha)(Sf)(\alpha) = 0$$

The main results of these papers were to identify some special classes of bounded analytic functions whose Toeplitz operators have commutants that exemplify the possible commutants of analytic Toeplitz operators.

That is, maybe there is a small set $\mathcal{S}$ of H^∞ functions so that for each ψ in H^∞, there is φ in $\mathcal{S}$ so that $\{T_\psi\}' = \{T_\varphi\}'$.

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It became clear that, inner functions and covering maps should be part of any such set $\mathcal{S}$ because Toeplitz operators associated with many other H^∞ functions have commutants the same as inner function or covering map Toeplitz operators.

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For example, the Fundamental Lemma, immediately implies

If φ and ψ are in H^∞ and there is an analytic function g so that $\varphi = g \circ \psi$, then $\{T_\varphi\}' \supset \{T_\psi\}'$.

So a natural question is: "If $\varphi = g \circ \psi$, when does $\{T_\varphi\}' = \{T_\psi\}'$?"

The main results of these papers were to identify some special classes of bounded analytic functions whose Toeplitz operators have commutants that exemplify the possible commutants of analytic Toeplitz operators.

Theorem: [C., 1978]

If ψ is a bounded analytic function on the disk $\mathbb{D}$

and α_0 is a point of the disk so that the inner factor of $\psi - \psi(\alpha_0)$

is a finite Blaschke product,

then there is a finite Blaschke product B so that

$$\{T_\psi\}' = \{T_B\}'$$

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$$\{T_\psi\}' = \{T_B\}'$$

In fact, the Blaschke product B is the "largest" inner function for which there is bounded function g so that $\psi = g \circ B$.

For B a finite Blaschke product of order n, except for $n(n-1)$ points of the disk for which $B(\alpha) = B(\beta)$ and $B'(\beta) = 0$,

$$\left(\left((B - B(\alpha))\, H^2\right)\right)^{\perp} = \text{ span } \{K_{\beta_1}, K_{\beta_2}, \cdots, K_{\beta_n}\}$$

where the points $\alpha = \beta_1$, β_2, $\cdots$, β_n are the n distinct points of $\mathbb{D}$ for which $B(\beta_j) = B(\alpha)$.

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The important fact behind this work is that the kernel functions K_α, $K_\alpha(z) = (1 - \overline{\alpha} z)^{-1}$ in H^2 and $K_\alpha(z) = (1 - \overline{\alpha} z)^{-2}$ in A^2, depend conjugate analytically on α, so if A is a linear operator so that AK_α is always in $\left((B - B(\alpha))\, H^2\right)^{\perp}$, then

$$AK_\alpha = \sum_j c_j K_{\beta_j}$$

where the c_j's and the K_{β_j}'s are conjugate analytic functions of α

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Observation:

For the study of commutants of Toeplitz operators, it is more important that a Blaschke product B is an n-to-1 map of $\mathbb{D}$ onto itself than the fact that T_B is a pure isometry on H^2.

Of course, since the points $\alpha = \beta_1, \beta_2, \cdots, \beta_n$ depend on α, we may write them as $\alpha = \beta_1(\alpha), \beta_2(\alpha), \cdots, \beta_n(\alpha)$.

In fact (!), if B is a finite Blaschke product of order n and α is a point of the disk that is *NOT* one of the $n(n-1)$ points of the disk for which

$B(\alpha) = B(\beta)$ and $B'(\beta) = 0$,

the maps $\alpha \mapsto \beta_j(\alpha)$ *are just the* n *branches of the analytic function* $B^{-1} \circ B$ *that is defined and arbitrarily continuable on the disk with the* $n(n-1)$ *exceptional points removed.*

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Theorem: (Cowen, 1974)

For B *a finite Blaschke product, the branches of* $B^{-1} \circ B$ *form a group whose normal subgroups are associated with compositional factorizations of* B *into compositions of two Blaschke products.*

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Fundamental Lemma:

For S a bounded operator on H^2 and ψ in H^∞, these $\bullet$ are equivalent

- *S commutes with T_ψ*
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Recall the **Fundamental Lemma:**

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Let $E_B = \{\alpha \in \mathbb{D} : B(\alpha) = B(\beta) \text{ for some } \beta \text{ with } B'(\beta) = 0\}$

be the *exceptional set for B.*

Use, W, the Riemann surface for $B^{-1} \circ B$ over $\mathbb{D} \setminus E_B$ to rewrite this as:

Fundamental Lemma(2):

Let B be a finite Blaschke product.

If S is a bounded operator on H^2, then S is in $\{T_B\}'$ if and only if

$$S^* K_\alpha = \sum_{j=1}^{n} c_j(\alpha) K_{\beta_j(\alpha)} \textit{ for each } \alpha \textit{ in } \mathbb{D} \setminus E_B.$$

We use this to write Sf as a function of α in the disk.

Theorem: (C., 1978). *Let B, E_B, and Riemann surface W be as above. If S is a bounded operator on H^2 that commutes with T_B, then there is a bounded analytic function G on the Riemann surface W so that for f in H^2,*

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta,\alpha))\beta'(\alpha)f(\beta(\alpha)) \tag{1}$$

where the sum is taken over the n branches of $B^{-1} \circ B$ at α. Moreover, if α_0 is a zero of order m of B', and ψ_1, ψ_2, $\cdots$, ψ_n is a basis for $\left((B - B(\alpha_0))\, H^2\right)^{\perp}$, then G has the property that

$$\sum G((\beta,\alpha))\beta'(\alpha)\psi_j(\beta(\alpha)) \text{ has a zero of order } m \text{ at } \alpha_0 \tag{2}$$

for $j = 1, 2, \cdots, n$.

Conversely, if G is a bounded analytic function on W that has property (2) at each zero of B', then (1) defines a bounded linear operator on H^2 with S in $\{T_B\}'$.

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Observation:

The class of 'multiple-valued weighted composition operators', an extension of classes of algebras of operators generated by multiplication and composition operators, appear to be useful in the study of certain kinds of problems in operator theory, including questions related to commutants.

Theorem: (C. & Wahl, 2012). *Let B, E_B, and W be as above.*

If S is a bounded operator on A^2 that commutes with T_B, then there is a bounded analytic function G on the Riemann surface W so that for f in A^2,

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha))\beta'(\alpha) f(\beta(\alpha)) \tag{3}$$

where the sum is taken over the n branches of $B^{-1} \circ B$ at α. Moreover, if α_0 is a zero of order m of B', and ψ_1, ψ_2, $\cdots$, ψ_n is a basis for $\left((B - B(\alpha_0))\, A^2\right)^{\perp}$, then G has the property that

$$\sum G((\beta, \alpha))\beta'(\alpha)\psi_j(\beta(\alpha)) \text{ has a zero of order } m \text{ at } \alpha_0 \tag{4}$$

for $j = 1, 2, \cdots, n$.

Conversely, if G is a bounded analytic function on W that has property (4) at each zero of B', then (3) defines a bounded linear operator on A^2 with S in $\{T_B\}'$.

Theorem: (C., 1978).

If B is a finite Blaschke product and S is a bounded operator on H^2

such that $ST_B = T_B S$,

then for all f in H^∞, Sf is also in H^∞.

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Theorem: (C. & Wahl, 2012).

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Theorem: (C. & Wahl, 2012).

If B is a finite Blaschke product and S is a bounded operator on A^2

such that $ST_B = T_BS$,

then for all f in H^∞, Sf is also in H^∞.

Ideas of the proof:

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha))\beta'(\alpha) f(\beta(\alpha)) \tag{3}$$

First, f is assumed to be bounded on the disk:

so $|f(\beta(\alpha))|$ is bounded by $\|f\|_\infty$.

Theorem: (C. & Wahl, 2012).

If B is a finite Blaschke product and S is a bounded operator on A^2

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Ideas of the proof:

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha))\beta'(\alpha) f(\beta(\alpha)) \tag{3}$$

Second, B is an n-to-1 map of the Riemann sphere to itself, so it has n poles outside the closed unit disk. In particular, B is analytic in a disk strictly larger than $\mathbb{D}$ and the $\beta'(\alpha)$ are bounded in a disk larger than $\mathbb{D}$.

Theorem: (C. & Wahl, 2012).

If B is a finite Blaschke product and S is a bounded operator on A^2

such that $ST_B = T_B S$,

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Ideas of the proof:

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta, \alpha))\beta'(\alpha) f(\beta(\alpha)) \tag{3}$$

Third, B' has $2n-2$ zeros on the Riemann sphere, $n-1$ in $\mathbb{D}$ and the other $n-1$ are reflections of these outside the closed unit disk.

In particular, B' is analytic and non-zero in an annulus strictly containing the unit circle.

This means $(B'(\alpha))^{-1}$ is bounded near the unit circle.

Theorem: (C. & Wahl, 2012).

If B is a finite Blaschke product and S is a bounded operator on A^2

such that $ST_B = T_B S$,

then for all f in H^∞, Sf is also in H^∞.

Ideas of the proof:

$$(Sf)(\alpha) = (B'(\alpha))^{-1} \sum G((\beta,\alpha))\beta'(\alpha) f(\beta(\alpha)) \tag{3}$$

Finally, the sum appears to depend on all n of the branches of $B^{-1} \circ B$ simultaneously.

Of course, it does, but using bounded analytic functions as multipliers, we can eliminate all but one term in the sum (3).

This allows us to show that each term of the sum $G((\beta,\alpha))$ is bounded separately and there are n bounded terms in the sum.

Theorem: (C. & Wahl, 2012).

If B is a finite Blaschke product and S is a bounded operator on A^2

such that $ST_B = T_BS$,

then for all f in H^∞, Sf is also in H^∞.

Corollary:

The commutants of T_B as an operator on H^2 and of T_B as an operator on A^2 are 'the same'.

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Corollary:

The commutants of T_B as an operator on H^2 and of T_B as an operator on A^2 are 'the same'.

The bounded analytic functions on the disk are dense in both H^2 and A^2. Since these functions are mapped in the same way as vectors in H^2 and A^2, the operators agree on all vectors common to H^2 and A^2.

These ideas apply in the same way to the weighted Bergman spaces as well.

Theorem: (C. & Wahl, 2012).

If B is a finite Blaschke product and S is a bounded operator on A^2

such that $ST_B = T_B S$,

then for all f in H^∞, Sf is also in H^∞.

Corollary:

The commutants of T_B as an operator on H^2 and of T_B as an operator on A^2 are 'the same'.

Corollary:

If ψ is a bounded analytic function on the disk $\mathbb{D}$

and α_0 is a point of the disk so that the inner factor of $\psi - \psi(\alpha_0)$

is a finite Blaschke product, there is finite Blaschke product B with

$$\{T_\psi\}' = \{T_B\}' \quad \textit{as operators on } A^2$$

Theorem: (C. & Wahl, 2012).

If B is a finite Blaschke product and S is a bounded operator on A^2 such that $ST_B = T_B S$, then for all f in H^∞, Sf is also in H^∞.

Corollary:

The commutants of T_B as an operator on H^2 and of T_B as an operator on A^2 are 'the same'.

Corollary:

If P is a bounded operator acting on H^2 such that $P^2 = P$ and $T_B P = P T_B$, then P is a bounded an operator acting on A^2 such that $P^2 = P$ and $T_B P = P T_B$.

The result

Corollary:

If P is a bounded operator acting on H^2 such that $P^2 = P$ and $T_B P = PT_B$, then P is a bounded an operator acting on A^2 such that $P^2 = P$ and $T_B P = PT_B$.

leads to some obvious, but still unsolved problems: "Which of the projections that commute with T_B on the Bergman space are self-adjoint?"

It is easy to see that many more self-adjoint projections commute with T_B on H^2 than on A^2 because multiplication by B is an isometry in H^2, but not on A^2.

The question "What is $\{T_B, T_B^*\}'$?" is largely unstudied!

Thank You!

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If $B(z) = z^2 \left(\dfrac{z - .5}{1 - .5z} \right)^2$, the group $B^{-1} \circ B$ is isomorphic to D_4.

D_4 has several normal subgroups, and most give trivial factorizations of B into the composition of a Blaschke product of order 1 and one of order 4. However, there is a normal subgroup that “finds” the non-trivial decomposition of B as $B = J_1 \circ J_2$ where $J_1(z) = z^2$ and $J_2(z) = z\dfrac{z - .5}{1 - .5z}$