Thoughts on Invariant Subspaces in Hilbert Spaces

Carl C. Cowen

IUPUI

and

Eva Gallardo Gutiérrez

Universidad Complutense de Madrid

***Celebration of Barbara's and Tom's Retirements*,**

Charlottesville, 12 October 2013

Congratulations Barbara and Tom!!

Thoughts on Invariant Subspaces in Hilbert Spaces

Carl C. Cowen

and

Eva Gallardo Gutiérrez

Thanks to: Plan Nacional I+D grant no. MTM2010-16679.

Speaker thanks the Departamento Análisis Matemático, Univ. Complutense de Madrid for hospitality during academic year 2012-13 and also thanks IUPUI for a sabbatical for last year that made this work possible.

Thank you for getting up so early!

Some Recent History:

We announced on 25 January that we had proved

the Invariant Subspace Theorem.

Thank you for getting up so early!

History:

We announced on 25 January that we had proved the Invariant Subspace Theorem.

On 26 January, we learned that we had **not** proved the Theorem.

Thank you for getting up so early!

Some Recent History:

We announced on 25 January that we had proved

the Invariant Subspace Theorem.

On 26 January, we learned we had **not** proved it.

On 12 June, we submitted a paper

including the main ideas of the earlier paper,

And on 14 August, we submitted another paper.

Thank you for getting up so early!

Some Recent History:

We announced on 25 January that we had proved

the Invariant Subspace Theorem.

On 26 January, we learned we had **not** proved it.

On 12 June, we submitted a paper

including the main ideas of the earlier paper,

And on 14 August, we submitted another paper.

Today, I'll talk about some of the history of the problem

and some of the results of these papers.

Some terminology:

If A is a bounded linear operator mapping a Banach space $\mathcal{X}$ into itself,

a closed subspace M of $\mathcal{X}$ is an *invariant subspace for A*

if for each v in M, the vector Av is also in M.

The subspaces $M = (0)$ and $M = \mathcal{X}$ are *trivial* invariant subspaces and we are not interested in these.

The *Invariant Subspace Question* is:

- Does every bounded operator on a Banach space have a non-trivial invariant subspace?

We will only consider vector spaces over the complex numbers.

If the dimension of the space $\mathcal{X}$ is finite and at least 2, then any linear transformation has eigenvectors and each eigenvector generates a one dimensional (non-trivial) invariant subspace.

The Jordan Canonical Form Theorem provides the information to construct all of the invariant subspaces of an operator on a finite dimensional space.

If A is an operator on $\mathcal{X}$ and x is a vector in $\mathcal{X}$, then the *cyclic subspace generated by* x is the closure of

$$\{ \, p(A)x : p \text{ is a polynomial } \}$$

Clearly, the cyclic subspace generated by x is an invariant subspace for A.

If the cyclic subspace generated by the vector x is all of $\mathcal{X}$,

we say x *is a cyclic vector for* A.

Every cyclic subspace is separable, in the sense of topology, so if $\mathcal{X}$ is *NOT* separable, every operator on $\mathcal{X}$ has non-trivial invariant subspaces.

Therefore, in thinking about the Invariant Subspace Question,

we restrict attention to infinite dimensional, separable Banach spaces.

Some history:

- Spectral Theorem for self-adjoint operators on Hilbert spaces gives

  invariant subspaces

- Beurling (1949): completely characterized the invariant subspaces of

  operator of multiplication by z on the Hardy Hilbert space, H^2

- von Neumann ('30's, 40's?), Aronszajn & Smith ('54):

  Every compact operator on a Banach space has invariant subspaces.

Some history:

- Spectral Theorem for self-adjoint operators
- Beurling (1949): invariant subspaces of isometric shift
- von Neumann ('30's, 40's?), Aronszajn & Smith ('54): compact operators
- Lomonosov ('73):

If S is an operator that commutes with an operator $T \neq \lambda I$,

and T commutes with a non-zero compact operator

then S has a non-trivial invariant subspace.

Some history:

- Spectral Theorem for self-adjoint operators
- Beurling (1949): invariant subspaces of isometric shift
- von Neumann ('30's, 40's?), Aronszajn & Smith ('54): compact operators
- Lomonosov ('73):

If S is an operator that commutes with an operator $T \neq \lambda I$,

and T commutes with a non-zero compact operator

then S has a non-trivial invariant subspace.

- Lomonosov did *not* solve ISP: Hadwin, Nordgren, Radjavi, Rosenthal('80)

Some history:

- Spectral Theorem for self-adjoint operators
- Beurling (1949): invariant subspaces of isometric shift
- von Neumann ('30's, 40's?), Aronszajn & Smith ('54): compact operators
- Lomonosov ('73): Yes, for S when $S \leftrightarrow T \leftrightarrow K$, if K compact
- Lomonosov did *not* solve ISP: Hadwin, Nordgren, Radjavi, Rosenthal('80)
- Enflo ('75/'87), Read ('85): Found operators on Banach spaces with only the trivial invariant subspaces!

Some history:

- Spectral Theorem for self-adjoint operators
- Beurling (1949): invariant subspaces of isometric shift
- von Neumann ('30's, 40's?), Aronszajn & Smith ('54): compact operators
- Lomonosov ('73): Yes, for S when $S \leftrightarrow T \leftrightarrow K$, if K compact
- Lomonosov did *not* solve ISP: Hadwin, Nordgren, Radjavi, Rosenthal('80)
- Enflo ('75/'87), Read ('85): Found operators on Banach spaces with only the trivial invariant subspaces!

The *(revised) Invariant Subspace Question* is:

- Does every bounded operator on a ~~Banach~~ Hilbert space have a non-trivial invariant subspace?

Rota's Universal Operators:

Defn: Let $\mathcal{X}$ be a Banach space, let U be a bounded operator on $\mathcal{X}$.

We say U is *universal for* $\mathcal{X}$ if for each bounded operator A on $\mathcal{X}$,

there is an invariant subspace M for U and a non-zero number λ

such that λA is similar to $U|_M$.

Rota's Universal Operators:

Defn: Let $\mathcal{X}$ be a Banach space, let U be a bounded operator on $\mathcal{X}$.

We say U is *universal for* $\mathcal{X}$ if for each bounded operator A on $\mathcal{X}$,

there is an invariant subspace M for U and a non-zero number λ

such that λA is similar to $U|_M$.

Rota proved in 1960 that if $\mathcal{X}$ is a separable, infinite dimensional Hilbert space, there are universal operators on $\mathcal{X}$!

Theorem (Caradus (1969))

If $\mathcal{H}$ is separable Hilbert space and U is bounded operator on $\mathcal{H}$ such that:

- The null space of U is infinite dimensional.
- The range of U is $\mathcal{H}$.

then U is universal for $\mathcal{H}$.

The Hardy Hilbert space on the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is:

$$H^2 = \{h \text{ analytic in } \mathbb{D} : h(z) = \sum_{n=0}^{\infty} a_n z^n \ \text{ with } \ \|h\|^2 = \sum |a_n|^2 < \infty\}$$

Isometry $z^n \leftrightarrow e^{in\theta}$ shows H^2 'is' subspace $\{h \in L^2(\partial\mathbb{D}) : h \sim \sum_{n=0}^{\infty} a_n e^{in\theta}\}$

H^2 is a *Hilbert space of analytic functions on* $\mathbb{D}$ in the sense that

for each α, the linear functional on H^2 given by $h \mapsto h(\alpha)$ is continuous.

Indeed, the inner product on H^2 gives $h(\alpha) = \langle h, K_\alpha \rangle$

where $K_\alpha(z) = (1 - \overline{\alpha} z)^{-1}$ for α in $\mathbb{D}$.

Consider four types of operators on H^2:

For f in $L^\infty(\partial\mathbb{D})$, *Toeplitz operator* T_f is operator given by $T_f h = P_+ fh$

where P_+ is the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2

For ψ a bounded analytic map of $\mathbb{D}$ into the complex plane,

the *analytic Toeplitz operator* T_ψ is

$$(T_\psi h)(z) = \psi(z)h(z) \quad \text{for } h \text{ in } H^2$$

Note: for ψ in H^∞, $\quad P_+\psi h = \psi h$

For φ an analytic map of $\mathbb{D}$ into itself, the *composition operator* C_φ is

$$(C_\varphi h)(z) = h(\varphi(z)) \quad \text{for } h \text{ in } H^2$$

and for ψ in H^∞ and φ an analytic map of $\mathbb{D}$ into itself,

the *weighted composition operator* $W_{\psi,\varphi} = T_\psi C_\varphi$ is

$$(W_{\psi,\varphi} h)(z) = \psi(z)h(\varphi(z)) \quad \text{for } h \text{ in } H^2$$

Lemma.

If f is a function in $H^\infty(\mathbb{D})$ and there is $\ell > 0$ so that $|f(e^{i\theta})| \geq \ell$ almost everywhere on the unit circle, then $1/f$ is in $L^\infty(\partial\mathbb{D})$ and the (non-analytic) Toeplitz operator $T_{1/f}$ is a left inverse for the analytic Toeplitz operator T_f.

Lemma.

If f is a function in $H^\infty(\mathbb{D})$ and there is $\ell > 0$ so that $|f(e^{i\theta})| \geq \ell$ almost everywhere on the unit circle, then $1/f$ is in $L^\infty(\partial\mathbb{D})$ and the (non-analytic) Toeplitz operator $T_{1/f}$ is a left inverse for the analytic Toeplitz operator T_f.

Theorem.

If f is a function in $H^\infty(\mathbb{D})$ for which there is $\ell > 0$ so that $|f(e^{i\theta})| \geq \ell$ almost everywhere on the unit circle and $Z_f = \{\alpha \in \mathbb{D} : f(\alpha) = 0\}$ is an infinite set, then the Toeplitz operator T_f^* is universal in the sense of Rota.

Lemma.

If f is a function in $H^\infty(\mathbb{D})$ and there is $\ell > 0$ so that $|f(e^{i\theta})| \geq \ell$ almost everywhere on the unit circle, then $1/f$ is in $L^\infty(\partial\mathbb{D})$ and the (non-analytic) Toeplitz operator $T_{1/f}$ is a left inverse for the analytic Toeplitz operator T_f.

Theorem.

If f is a function in $H^\infty(\mathbb{D})$ for which there is $\ell > 0$ so that $|f(e^{i\theta})| \geq \ell$ almost everywhere on the unit circle and $Z_f = \{\alpha \in \mathbb{D} : f(\alpha) = 0\}$ is an infinite set, then the Toeplitz operator T_f^* is universal in the sense of Rota.

Proof:

By the Lemma, the analytic Toeplitz operator T_f has a left inverse, so the Toeplitz operator T_f^* has a right inverse and T_f^* maps $H^2(\mathbb{D})$ onto itself. Since $T_f^*(K_\alpha) = \overline{f(\alpha)}K_\alpha = 0$ for α in Z_f, the kernel of T_f^* is infinite dimensional. Thus, Caradus' Theorem implies T_f^* is universal. ∎

Some previously known Universal Operators (in sense of Rota):

Best Known: adjoint of a unilateral shift of infinite multiplicity:

If S is analytic Toeplitz operator whose symbol is an inner function that is *not* a finite Blaschke product, then S^* is a universal operator.

Some previously known Universal Operators (in sense of Rota):

Also well known (Nordgren, Rosenthal, Wintrobe ('84,'87)):

If φ is an automorphism of $\mathbb{D}$ with fixed points ± 1 and Denjoy-Wolff point 1, that is, $\varphi(z) = \dfrac{z+s}{1+sz}$ for $0 < s < 1$,

then a translate of the composition operator C_φ is a universal operator.

Some previously known Universal Operators (in sense of Rota):

Also well known (Nordgren, Rosenthal, Wintrobe ('84,'87)):

If φ is an automorphism of $\mathbb{D}$ with fixed points ± 1 and Denjoy-Wolff point 1, that is, $\varphi(z) = \dfrac{z+s}{1+sz}$ for $0 < s < 1$, then a translate of the composition operator C_φ is a universal operator.

In 2011, C. and Gallardo Gutiérrez showed that this translate, restricted to a co-dimension one invariant subspace on which it is universal, is unitarily equivalent to the adjoint of the analytic Toeplitz operator T_ψ where ψ is a translate of the covering map of the disk onto interior of the annulus $\sigma(C_\varphi)$.

Some previously known Universal Operators (in sense of Rota):

Also well known (Nordgren, Rosenthal, Wintrobe ('84,'87)):

If φ is an automorphism of $\mathbb{D}$ with fixed points ± 1 and Denjoy-Wolff point 1, that is, $\varphi(z) = \dfrac{z+s}{1+sz}$ for $0 < s < 1$,
then a translate of the composition operator C_φ is a universal operator.

In 2011, C. and Gallardo Gutiérrez showed that this translate, restricted to a co-dimension one invariant subspace on which it is universal, is unitarily equivalent to the adjoint of the analytic Toeplitz operator T_ψ where ψ is a translate of the covering map of the disk onto interior of the annulus $\sigma(C_\varphi)$.

In C.'s thesis ('76): The analytic Toeplitz operators S and T_ψ *DO NOT* commute with non-trivial compact operators.

Also proved: *IF* an analytic Toeplitz operator commutes with a non-trivial compact, then the compact operator is quasi-nilpotent.

Some previously known Universal Operators (in sense of Rota):

Main Theorem of June paper. (C. and Gallardo Gutiérrez, 2013)

There are bounded analytic functions φ and ψ on the unit disk

and an analytic map J of the unit disk into itself

so that the Toeplitz operator T_φ^* is a universal operator in the sense of Rota

and the weighted composition operator $W_{\psi,J}^*$

is an injective, compact operator with dense range

that commutes with the universal operator T_φ^*.

Let $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z^2 > -1 \text{ and } \operatorname{Re} z < 0\}$,

the region in second quadrant above branch of the hyperbola $2xy = -1$.

Let σ be the Riemann map of $\mathbb{D}$ onto Ω defined by

$$\sigma(z) = \frac{-1+i}{\sqrt{z+1}}$$

where $\sqrt{\cdot}$ is the branch on the halfplane $\{z : \operatorname{Re} z > 0\}$ satisfying $\sqrt{1} = 1$.

Notice that $\sigma(1) = (-1+i)/\sqrt{2}$, $\sigma(0) = -1+i$, and $\sigma(-1) = \infty$.

We define φ on the unit disk by

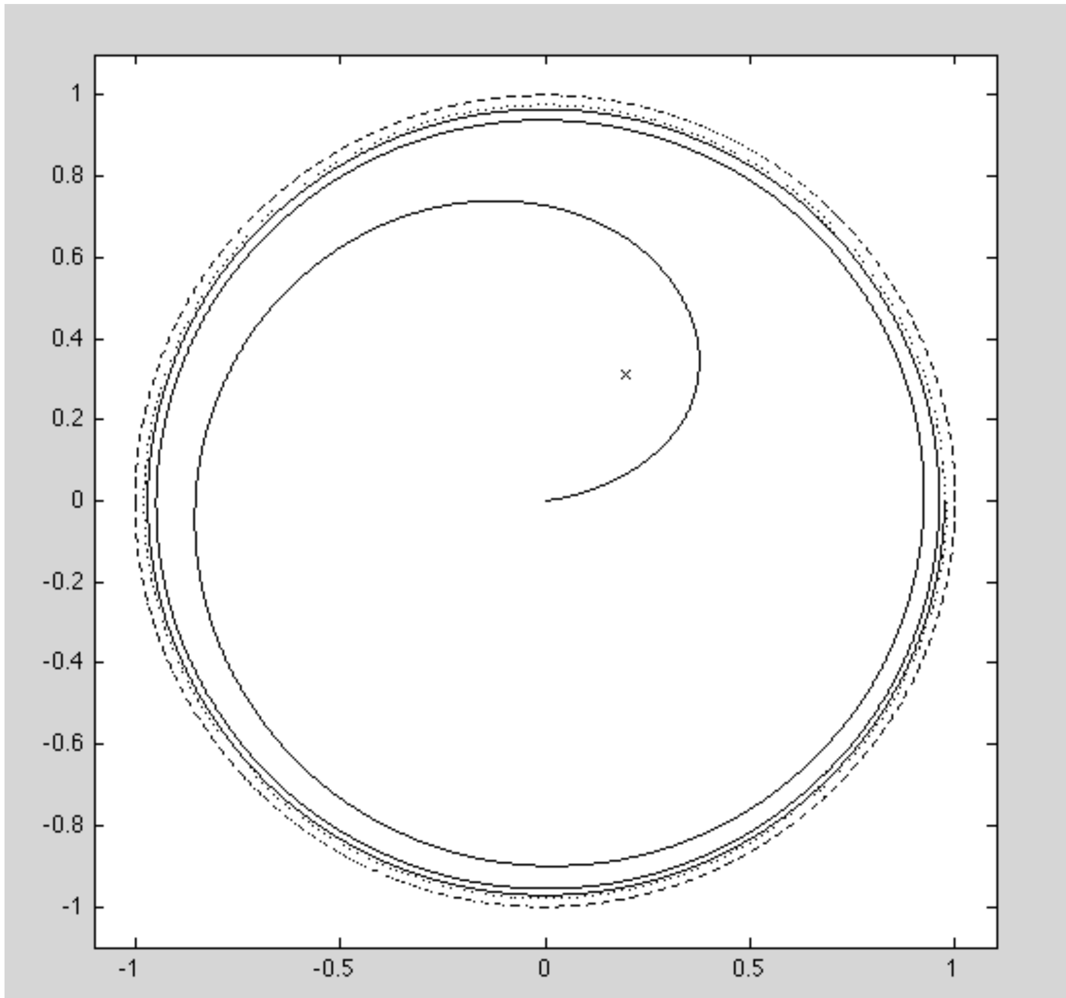
$$\varphi(z) = e^{\sigma(z)} - e^{\sigma(0)} = e^{\sigma(z)} - e^{-1+i}$$

The function e^σ maps the curve $\Gamma = \{e^{i\theta} : -\pi < \theta < \pi\}$,

the unit circle except -1, onto curve spiraling out from origin to $\partial\mathbb{D}$.

Each circle of radius r intersects curve $e^{\sigma(\Gamma)}$ in exactly one point.

Closure $e^{\sigma(\Gamma)}$ is the set $\{0\} \cup e^{\sigma(\Gamma)} \cup \partial\mathbb{D}$ and distance $e^{\sigma(0)}$ to $e^{\sigma(\Gamma)} > 0$.



Let J be the analytic map of the unit disk into itself given by

$$J(z) = \sigma^{-1}(\sigma(z) + 2\pi i)$$

Letting $\psi(z) = (z+1)/2$, since $\psi(-1) = 0$, it follows that $W_{\psi,J}$ is a compact weighted composition operator, and since $\varphi \circ J = \varphi$, we see that $W_{\psi,J}$ commutes with T_φ.

Observations:

- The best known operators that are universal in the sense of Rota are, or are unitarily equivalent to, adjoints of analytic Toeplitz operators.

- Some universal operators commute with compact operators and some do not.

Our Goals Today:

- There are *VERY MANY* analytic Toeplitz operators

  whose adjoints are universal operators in the sense of Rota

  and *VERY MANY* of them commute with non-trivial compact operators!

Our Goals Today:

- There are *VERY MANY* analytic Toeplitz operators

  whose adjoints are universal operators in the sense of Rota

  and *VERY MANY* of them commute with non-trivial compact operators!

- Describe some properties of such operators

Our Goals Today:

- There are *VERY MANY* analytic Toeplitz operators

  whose adjoints are universal operators in the sense of Rota

  and *VERY MANY* of them commute with non-trivial compact operators!

- Describe some properties of such operators

- Raise two questions about invariant subspaces of 'the' Shift Operator

  that we haven't been able to answer.

Let $\mathcal{U}_0$ be the set of adjoints of analytic Toeplitz operators that the Lemma above implies are left invertible, that is

$$\mathcal{U}_0 = \{T_f^* : f \in H^\infty \text{ and } 1/f \in L^\infty(\partial\mathbb{D})\}$$

and let

$$\mathcal{U} = \{T_f^* \in \mathcal{U}_0 : \text{ kernel}(T_f^*) \text{ is infinite dimensional } \}$$

Let $\mathcal{U}_0$ be the set of adjoints of analytic Toeplitz operators that the Lemma above implies are left invertible, that is

$$\mathcal{U}_0 = \{T_f^* : f \in H^\infty \ \text{ and } \ 1/f \in L^\infty(\partial\mathbb{D})\}$$

and let

$$\mathcal{U} = \{T_f^* \in \mathcal{U}_0 : \ \text{kernel}(T_f^*) \text{ is infinite dimensional } \}$$

Theorem.

If f is in H^∞ and T_f^* is in $\mathcal{U}$, the Toeplitz operator T_f^* is universal for H^2.

Let $\mathcal{U}_0$ be the set of adjoints of analytic Toeplitz operators that the Lemma above implies are left invertible, that is

$$\mathcal{U}_0 = \{T_f^* : f \in H^\infty \text{ and } 1/f \in L^\infty(\partial\mathbb{D})\}$$

and let

$$\mathcal{U} = \{T_f^* \in \mathcal{U}_0 : \text{ kernel}(T_f^*) \text{ is infinite dimensional } \}$$

Theorem.

If f is in H^∞ and T_f^* is in $\mathcal{U}$, the Toeplitz operator T_f^* is universal for H^2.

Corollary.

If f and g are in H^∞ with T_f^* in $\mathcal{U}$ and T_g^* in $\mathcal{U}_0$,

then $T_f^* T_g^* = T_{fg}^*$ is also in $\mathcal{U}$ and is a universal operator for H^2.

For F bounded on H^2, the commutant of F is the closed algebra in $\mathcal{B}(H^2)$

$$\{F\}' = \{G \in \mathcal{B}(H^2) : GF = FG\}$$

For f in H^∞, clearly $\{T_f^*\}'$ includes T_g^* for all g in H^∞.

Definition. For T_f^* in $\mathcal{U}$, let $\mathcal{C}_f$ be the set of compact operators in $\{T_f^*\}'$:

$$\mathcal{C}_f = \{G \in \mathcal{B}(H^2) : G \text{ is compact, and } T_f^* G = G T_f^*\}$$

For F bounded on H^2, the commutant of F is the closed algebra in $\mathcal{B}(H^2)$

$$\{F\}' = \{G \in \mathcal{B}(H^2) : GF = FG\}$$

For f in H^∞, clearly $\{T_f^*\}'$ includes T_g^* for all g in H^∞.

Definition. For T_f^* in $\mathcal{U}$, let $\mathcal{C}_f$ be the set of compact operators in $\{T_f^*\}'$:

$$\mathcal{C}_f = \{G \in \mathcal{B}(H^2) : G \text{ is compact, and } T_f^* G = G T_f^*\}$$

Theorem.

Let T_f^* be in $\mathcal{U}$. The set $\mathcal{C}_f$ is a closed ideal in $\{T_f^*\}'$ and, in particular, g and h in H^∞ and G in $\mathcal{C}_f$ implies $T_g^* G$, $G T_h^*$, and $T_g^* G T_h^*$ are all in $\mathcal{C}_f$.

Moreover, *every* operator in $\mathcal{C}_f$ is quasi-nilpotent.

For some T_f^* in $\mathcal{U}$, including all the classical universal operators noted above, the algebra $\mathcal{C}_f$ is $\{0\}$.

On the other hand, for many operators T_f^* in $\mathcal{U}$, including the example T_φ^* from our earlier paper, the algebra $\mathcal{C}_f$ is quite large!!

For some T_f^* in $\mathcal{U}$, including all the classical universal operators noted above,

the algebra $\mathcal{C}_f$ is $\{0\}$.

On the other hand, for many operators T_f^* in $\mathcal{U}$, including the example T_φ^* from our earlier paper, the algebra $\mathcal{C}_f$ is quite large!!

Following is a trivial, but surprising, application of Lomonosov's theorem:

Theorem. (!!)

If f is a non-constant function in H^∞ for which $\mathcal{C}_f \neq \{0\}$,

there is a backward shift invariant subspace,

$L = (\eta H^2)^\perp$ for some inner function η,

that is invariant for every operator in $\{T_f^*\}'$.

For some T_f^* in $\mathcal{U}$, including all the classical universal operators noted above, the algebra $\mathcal{C}_f$ is $\{0\}$.

On the other hand, for many operators T_f^* in $\mathcal{U}$, including the example T_φ^* from our earlier paper, the algebra $\mathcal{C}_f$ is quite large!!

Following is a trivial, but surprising, application of Lomonosov's theorem:

Theorem. (!!)

If f is a non-constant function in H^∞ for which $\mathcal{C}_f \neq \{0\}$,

there is a backward shift invariant subspace,

$L = (\eta H^2)^\perp$ for some inner function η,

that is invariant for every operator in $\{T_f^*\}'$.

Proof: T_z^* commutes with T_f^*.

Theorem. (!!)

If f is a non-constant function in H^∞ for which $\mathcal{C}_f \neq \{0\}$,

there is a backward shift invariant subspace,

$L = (\eta H^2)^\perp$ for some inner function η,

that is invariant for every operator in $\{T_f^*\}'$.

In the case of the T_φ^* and the compact operator $W_{\psi,J}^*$ noted above,

the commutant $\{T_\varphi^*\}'$ is known!

Theorem. (!!)

If f is a non-constant function in H^∞ for which $\mathcal{C}_f \neq \{0\}$,

there is a backward shift invariant subspace,

$L = (\eta H^2)^\perp$ for some inner function η,

that is invariant for every operator in $\{T_f^*\}'$.

In the case of the T_φ^* and the compact operator $W_{\psi,J}^*$ noted above,

the commutant $\{T_\varphi^*\}'$ is known!

It is the algebra generated by T_z^* and C_J^*!

To prove the Invariant Subspace Theorem, need to show that every bounded operator, A, on H^2 has an invariant subspace. But the universality of T_f^* in $\mathcal{U}$ means that we are interested only in restrictions of T_f^* to its infinite dimensional invariant subspaces, M. This means the Invariant Subspace Theorem will be proved if every infinite dimensional invariant subspace, M, for T_f^* contains a smaller subspace that is also invariant for T_f^*.

Our strategy for applying universal Toeplitz operators to the Invariant Subspace Problem is to also consider operators that commute with the universal operator.

Theorem.

Let T be a universal operator on H^2 that is in the class $\mathcal{U}$, and let M be an infinite dimensional, proper invariant subspace for T. If W is an operator on H^2 that commutes with T, then either $\text{kernel}\,(W) \cap M = (0)$, or $M \subset \text{kernel}\,(W)$, or $\text{kernel}\,(W) \cap M$ is a proper subspace of M that is invariant for T.

Our strategy for applying universal Toeplitz operators to the Invariant Subspace Problem is to also consider operators that commute with the universal operator.

Theorem.

Let T be a universal operator on H^2 that is in the class $\mathcal{U}$,

and let M be an infinite dimensional, proper invariant subspace for T.

If W is an operator on H^2 that commutes with T, then

either $\mathrm{kernel}\,(W) \cap M = (0)$, or $M \subset \mathrm{kernel}\,(W)$,

or $\mathrm{kernel}\,(W) \cap M$ is a proper subspace of M that is invariant for T.

Proof. If $TW = WT$, then for x in $\mathrm{kernel}\,(W)$, we have

$$W(Tx) = T(Wx) = T0 = 0$$

so Tx is also in $\mathrm{kernel}\,(W)$ and $\mathrm{kernel}\,(W)$ is an invariant subspace for T.

Theorem.

Let T be a universal operator on H^2 that is in the class $\mathcal{U}$,

and let M be an infinite dimensional, proper invariant subspace for T.

If W is an operator on H^2 that commutes with T, then

either $\text{kernel}\,(W) \cap M = (0)$, or $M \subset \text{kernel}\,(W)$,

or $\text{kernel}\,(W) \cap M$ is a proper subspace of M that is invariant for T.

Corollary.

Let M be an infinite dimensional, proper invariant subspace for T,

a universal operator on H^2 that is in the class $\mathcal{U}$.

If M contains a vector, $v \neq 0$, that is non-cyclic vector for the backward shift

and η is smallest inner function for which $T_\eta^* v = 0$, then $M \subset \text{kernel}\,(T_\eta^*)$,

or else $\text{kernel}\,(T_\eta^*) \cap M$ is a non-trivial invariant subspace for T.

This suggests the question

> *Does every closed, infinite dimensional subspace of H^2 include a non-zero, non-cyclic vector for the backward shift?*

but Prof. N. Nikolski pointed out that the answer to this question is “No!”.

This suggests the question

> *Does every closed, infinite dimensional subspace of H^2 include a non-zero, non-cyclic vector for the backward shift?*

but Prof. N. Nikolski pointed out that the answer to this question is "No!".

On the other hand, we are not interested in arbitrary subspaces of H^2 and we specialize our query to address the issue at hand:

Question 1: *Does every closed, infinite dimensional subspace of H^2 that is a proper invariant subspace for an operator in the class $\mathcal{U}$ include a non-zero vector that is not cyclic for the backward shift?*

The other alternative in the Corollary above is that $M \subset \text{kernel}\,(T_\eta^*)$ and *every* vector in M is non-cyclic for the backward shift! Thus, we have

Corollary.

If M is an infinite dimensional, proper invariant subspace for T, a universal operator on H^2 that is in the class $\mathcal{U}$ and M contains a vector, $v \neq 0$, that is not cyclic for the backward shift and also a vector w that is cyclic for the backward shift, then, for η the smallest inner function for which $T_\eta^* v = 0$, the subspace $\text{kernel}\,(T_\eta^*) \cap M$ is a proper subspace of M that is invariant for T.

On the other hand, another possible reduction for this situation leads to the following question:

Question 2: *Suppose M is an infinite dimensional closed subspace that is invariant for T, a universal operator in the class $\mathcal{U}$, and suppose η is an inner function for which $M \subset kernel(T_\eta^*)$.*

Is there always an inner function ζ dividing η so that $(0) \neq M \cap kernel(T_\zeta^*) \neq M$?

On the other hand, another possible reduction for this situation leads to the following question:

Question 2: *Suppose M is an infinite dimensional closed subspace that is invariant for T, a universal operator in the class $\mathcal{U}$, and suppose η is an inner function for which $M \subset kernel(T_\eta^*)$.*

Is there always an inner function ζ dividing η so that $(0) \neq M \cap kernel(T_\zeta^*) \neq M$?

If the answers to both Question 1 and Question 2 are 'Yes',

then every bounded operator on a Hilbert space of dimension at least 2

has a non-trivial invariant subspace!

THANK YOU!