

Universal quantum computation with weakly integral anyons

Shawn X. Cui¹ · Seung-Moon Hong² ·
Zhenghan Wang^{1,3}

Received: 23 February 2014 / Accepted: 30 April 2015 / Published online: 12 May 2015
© Springer Science+Business Media New York 2015

Abstract Harnessing non-abelian statistics of anyons to perform quantum computational tasks is getting closer to reality. While the existence of universal anyons by braiding alone such as the Fibonacci anyon is theoretically a possibility, accessible anyons with current technology all belong to a class that is called weakly integral—anyons whose squared quantum dimensions are integers. We analyze the computational power of the first non-abelian anyon system with only integral quantum dimensions— $D(S_3)$, the quantum double of S_3 . Since all anyons in $D(S_3)$ have finite images of braid group representations, they cannot be universal for quantum computation by braiding alone. Based on our knowledge of the images of the braid group representations, we set up three qutrit computational models. Supplementing braidings with some measurements and ancillary states, we find a universal gate set for each model.

Keywords Anyonic quantum computation · Universal gate set · Braid group

The third author thanks A. Kitaev for explaining his encoding. S.X.C. and Z.W. are partially supported by NSF DMS 1108736.

✉ Zhenghan Wang
zhenghwa@math.ucsb.edu; zhenghwa@microsoft.com

Shawn X. Cui
xingshan@math.ucsb.edu

Seung-Moon Hong
SeungMoon.Hong@utoledo.edu

¹ Department of Mathematics, University of California, Santa Barbara, CA 93106, USA

² Department of Mathematics and Statistics, University of Toledo, Toledo, OH 43606, USA

³ Microsoft Research, Station Q, University of California, Santa Barbara, CA 93106, USA

1 Introduction

Harnessing non-abelian statistics of anyons to perform quantum computational tasks is getting closer to reality. While the existence of universal anyons by braiding alone such as the Fibonacci anyon is theoretically a possibility [15, 19], accessible anyons with current technology all belong to a class that is called weakly integral (WI)—anyons whose squared quantum dimensions are integers. A famous WI anyon is the Ising anyon σ with $d_\sigma = \sqrt{2}$, which is believed to model the non-abelian quasi-particle in the fractional quantum Hall liquids at $\nu = \frac{5}{2}$ [17]. Other WI anyons with Property F [16] include the metaplectic anyons [6, 7] and those in quantum double of finite groups [5]. Certain topological defects or ends of 1D nanowires also behave as WI anyons [2, 10]. It is conjectured that all WI anyons have finite images of braid group representations [16], if so then they cannot be universal for quantum computation by braiding alone.

In this paper, we analyze the computational power of the anyon system $D(\mathcal{S}_3)$ —the quantum double of \mathcal{S}_3 . $D(\mathcal{S}_3)$ is the first non-abelian anyon system with only integral quantum dimensions [4]. There are 8 anyon types in the theory, which are denoted by A, B, C, D, E, F, G, H as in [3] with quantum dimensions $\{1, 1, 2, 3, 3, 2, 2, 2\}$. It is known that all braid images are finite [5]. It follows that to obtain a universal quantum gate set, we have to go beyond braiding. The natural extra resources are measurements and ancillary states. Using measurement to gain extra computational power is tricky because universal quantum computation can be performed by measurement alone. Similar caution applies to ancillary states as cluster state quantum computation shows. Therefore, we have to be careful in choosing physically reasonable extra resources from measurements and ancillary states.

In [9] (see also [13]), the anyon of quantum dimension $d_D = 3$, denoted as D , is made universal for quantum computation by encoding a qutrit in the A, C fusion channels of a pair of D anyons. In the usual anyonic quantum computing model, two fusion channels such as A, C of a pair of D anyons would be used to encode a qubit instead of a qutrit, but the authors split the single fusion channel C into a two-dimensional internal computational space because the anyon C is the two-dimensional representation of \mathcal{S}_3 . Since $D(\mathcal{S}_3)$ is a discrete gauge theory, this encoding is justified on physical ground and computation is performed using only the color (topological) degrees of freedom of anyons. Universality based on similar encodings for other finite groups is established in [14, 18]. In this paper, we follow the usual scheme in anyonic quantum computation by encoding information in the subspace of anyonic fusion tree basis without splitting any fusion channels, i.e., the computational subspace is spanned by basis elements from labeling a single fusion tree. Since two pairs of D anyons has 9 fusion channels, we have many choices of encoding a qutrit by choosing a three-dimensional fusion subspace. Based on our analysis of the representations of the braid groups, we propose three different encodings of qutrits: one with 3 fusion tree bases, and two with superpositions of fusion tree bases. It is possible, as suggested by Kitaev to the third author, that the splitting of the fusion channel C as in [9] can be understood as a non-local encoding using superpositions of different fusion trees. Note that in the encoding in [9], there is a bureau of standards, which is mathematically a based frame.

The contents of the paper is as follows. In Sect. 2, we provide the detail of our adaptive anyonic quantum computing models and prove that a qutrit gate set convenient for our purpose is universal. There are three natural choices in our setup that are called U -model, V -model and W -model, respectively. We also define the measurements and ancillary states that we are going to use later. In Sect. 3, we prove that the U - and V -models are universal when braidings are supplemented by the two measurements defined in Sect. 2, and the W -model needs the extra ancillary state to become universal. Our major technical advance is organized into two appendices. We obtained complete solutions of all modular categories with the same fusion rules as $D(\mathcal{S}_3)$. But to save space, we list in “Appendix 1” only the complete data for the category $D(\mathcal{S}_3)$. For this paper, any other theory will work equally well. The complete list of data in “Appendix 1” is used in “Appendix 2” to analyze the images of the braid group representations of \mathcal{B}_4 . We give complete information of the finite images as abstract groups and as concretes matrices with respect to the computational bases of our models. These matrices are braiding quantum circuits of our models. Many interesting finite groups such as the Hessian group of order 216 appeared as images of braid group representations.

2 Adaptive anyonic quantum computing model

A pure anyonic quantum computing model as illustrated by Fig. 7.1 in [21] is to implement a circuit by braiding alone. Measurement is only done in the end by fusing anyons together. In particular, we are not allowed to do measurements in the middle of the computation. Unlike the standard circuit model, a computation with measurements during the computing process is not always equivalent to one that all measurements are postponed to the end of the computation. Since WI anyons provide only very limited circuits by braiding alone, we have to rely on other resources to obtain a universal gate set. The obvious places to look for are measurements in the middle of the computation and ancillary states. Since measurements of anyon charges beyond fusing two anyons are subtle, we want to do as little measurement as possible so that we do not decohere or leak the protected information in the computational subspace. In this section, we use $D(\mathcal{S}_3)$ to illustrate such adaptive models that braiding gates are supplemented with measurements and ancillary states. Our goal is to find minimal extra resources beyond braidings to obtain a universal gate set. Therefore, though important for physically realizing the extra resources, we will not justify our choices of measurements or ancillary states physically. Another important issue that we did not address is the issue of leakage. We think there is no damaging leakage in our model, but will leave a careful analysis to the future.

2.1 The integral anyon system $D(\mathcal{S}_3)$

The irreducible representations (irreps) of the quantum group $D(\mathcal{S}_3)$, called the Drinfeld double or quantum double of \mathcal{S}_3 , correspond to pairs (C, ρ_u) , where C is a

conjugacy class of \mathcal{S}_3 , u an element in C and ρ_u an irrep of the centralizer of u in \mathcal{S}_3 . For a fixed conjugacy class C , the corresponding irreps of $D(\mathcal{S}_3)$ do not depend on the choice of the element u . There are three conjugacy classes of \mathcal{S}_3 , namely $C_1 = \{e\}$, $C_2 = \{(12), (23), (13)\}$, $C_3 = \{(123), (132)\}$. For C_1 , the centralizer of e is \mathcal{S}_3 , which has three irreps, i.e., the trivial one, the sign one and the two-dimensional one. We denote them by A , B and C , respectively. For C_2 , we pick (12) and its centralizer is isomorphic to \mathbb{Z}_2 , which has two irreps. We denote the trivial one by D and the other one by E . For C_3 , the centralizer of (123) is isomorphic to \mathbb{Z}_3 , which has three irreps, all of which are one-dimensional. More precisely, they correspond to mapping the generator of \mathbb{Z}_3 to 1 , $\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$ and ω^2 . We denote these three irreps by F , G and H , respectively. Therefore, there are in total 8 irreps of $D(\mathcal{S}_3)$. As a unitary modular category, $Rep(D(\mathcal{S}_3))$ has 8 isomorphism classes of simple objects. Since simple objects in unitary modular categories model anyons, we also call them anyon types. The 8 anyon types were denoted by A, B, C, D, E, F, G and H . In the discrete gauge theory, an anyon of type A is called the vacuum; anyons of types B and C are purely electric charges; anyons of types D and F are purely magnetic fluxes; while anyons of types E, G and H are dyons. We will not always distinguish between anyon types (isomorphism classes of simple objects) and anyons (simple objects) carefully because for $D(\mathcal{S}_3)$ this distinction will not make any difference. Detailed explanations of the quantum double of \mathcal{S}_3 can be found in many references, e.g., [3, 11].

We list the irreps of $D(\mathcal{S}_3)$ and their quantum dimensions in Table 1 and the fusion rules in Table 2.

Table 1 $D(\mathcal{S}_3)$

Flux (conjugacy class)	Centralizer	Charge	qdim
$C_1 = \{e\}$	$Z(e) = \mathcal{S}_3$	$1, -1, 2$	$1, 1, 2$
$C_2 = \{(12), (13), (23)\}$	$Z((12)) = \mathbb{Z}_2 = \{e, (12)\}$	$+, -$	$3, 3$
$C_3 = \{(123), (132)\}$	$Z((123)) = \mathbb{Z}_3 = \{e, (123), (132)\}$	$1, \omega, \bar{\omega}$	$2, 2, 2$

Table 2 Fusion rules of $D(\mathcal{S}_3)$

\otimes	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	C	E	D	F	G	H
C	C	C	$A \oplus B \oplus C$	$D \oplus E$	$D \oplus E$	$G \oplus H$	$F \oplus H$	$F \oplus G$
D	D	E	$D \oplus E$	$A \oplus C \oplus F \oplus G \oplus H$	$B \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
E	E	D	$D \oplus E$	$B \oplus C \oplus F \oplus G \oplus H$	$A \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
F	F	F	$G \oplus H$	$D \oplus E$	$D \oplus E$	$A \oplus B \oplus F$	$H \oplus C$	$G \oplus C$
G	G	G	$F \oplus H$	$D \oplus E$	$D \oplus E$	$H \oplus C$	$A \oplus B \oplus G$	$F \oplus C$
H	H	H	$F \oplus G$	$D \oplus E$	$D \oplus E$	$G \oplus C$	$F \oplus C$	$A \oplus B \oplus H$

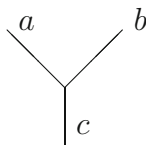
The modular representation of $SL(2, \mathbb{Z})$ is given by:

$$T = \text{diag}(1, 1, 1, -1, 1, 1, \omega, \omega^2).$$

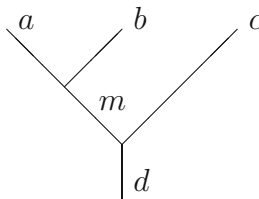
$$S = \frac{1}{6} \begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\ 1 & 1 & 2 & -3 & -3 & 2 & 2 & 2 \\ 2 & 2 & 4 & 0 & 0 & -2 & -2 & -2 \\ 3 & -3 & 0 & 3 & -3 & 0 & 0 & 0 \\ 3 & -3 & 0 & -3 & 3 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 & 0 & 4 & -2 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & -2 & 4 \\ 2 & 2 & -2 & 0 & 0 & -2 & 4 & -2 \end{pmatrix}$$

2.2 Fusion tree basis

An anyon c can split into a pair of anyons (a, b) if the triple (a, b, c) is admissible [21]. We denote this process by

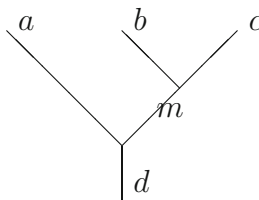


The anyon a can continue to split into another pair of anyons. Consider the following splitting tree:



There is a Hilbert space V_d^{abc} for the 4 anyons a, b, c, d , where the labeled splitting trees with choices of anyon m that make the splitting tree admissible at each trivalent vertex form a basis of $V_d^{a,b,c}$. We imagine the splitting process as going from the bottom to the top and then the fusing process going from the top to the bottom. Therefore, we will often also refer to a splitting tree as a fusion tree.

We also have another splitting tree.

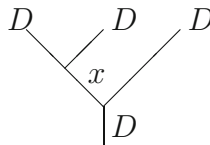


The labeled fusion trees provide another basis for the same Hilbert space V_d^{abc} . Hence, there is a transformation matrix F that relates these two bases, which is called an F -matrix.

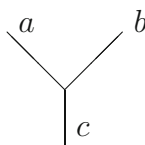
$$\begin{array}{c} a & & b & & c \\ & \diagdown & / & & / \\ & & m & & \\ & & | & & \\ & & d & & \end{array} = \sum_n F_{d;nm}^{abc} \begin{array}{c} a & & b & & c \\ & \diagdown & / & & / \\ & & n & & \\ & & | & & \\ & & d & & \end{array}$$

Assume that the fusion tree on the left-hand side of the above equation is the j_m -th basis element and the fusion tree on the right-hand side is the i_n -th basis element in the other basis. Then $F_{d;nm}^{abc}$ will denote the (i_n, j_m) -entry of F_d^{abc} when the theory has no multiplicities in the fusion rules such as $D(S_3)$. The numerical values $F_{d;nm}^{abc}$ are called 6j symbols. In the following, we always assume that there are no multiplicities in the fusion rules, i.e., the fusion coefficients are either 0 or 1. The matrices F_d^{abc} can be chosen to be unitary for a unitary theory.

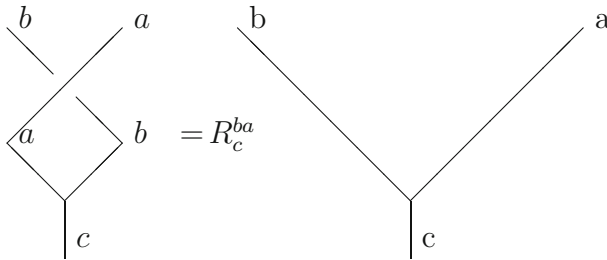
Remark 1 For the case of $D(S_3)$, recall that the eight anyon types are denoted by A, B, C, D, E, F, G ; and H . For some reason, we order the basis elements in the order $A, B, G, D, E, F, C, \text{ and } H$. This is the order we use to compute the F -matrices in “Appendix 1.” For example, the basis for the following fusion tree is $\{A, G, F, C, H\}$.



The Hilbert space V_c^{ab} associated with the following splitting tree is one-dimensional.



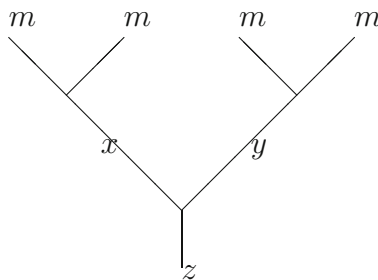
Braiding the two anyons a, b corresponds to a unitary transformation B_c^{ab} from V_c^{ab} to V_c^{ba} . The image of the fusion tree basis of V_c^{ab} under B_c^{ab} is a scalar multiple of the chosen fusion tree basis of V_c^{ba} . This scalar is denoted by R_c^{ba} , which is called the R -symbol.



The $6j$ -symbols and R -symbols are the data needed to compute the amplitudes of creating anyons from the vacuum, braiding some of them and then fusing them back to the vacuum. The approximation of these probabilities for such processes is the output of anyonic quantum computational models. Similar orthonormal basis exists for any Hilbert space $V_x^{ab\dots c}$. Protected information is encoded into some subspaces of $V_x^{ab\dots c}$, which are called computational subspaces. There are no canonical choices of computational subspaces.

2.3 Encoding of qudits

Consider the following fusion tree:



Namely, we start with an anyon of type z and split it into 4 anyons, all of which have the same anyon type m . All the pairs of (x, y) that make the above splitting tree admissible form a natural basis of V_z^{mmmm} . We denote them by $\{|x, y; m, z\rangle\}$. When there is no confusion, we will use the abbreviation $|xy\rangle$ for $|x, y; m, z\rangle$. We use this basis of the Hilbert space V_z^{mmmm} or linear combinations of some of them as our computational basis for a 1-qudit. For a particular theory, this Hilbert space is not big enough for all qudits, but usually we are only interested in a qubit or a qutrit.

To carry out computation, we can braid the first anyon m with the second, the second with the third and the third with the fourth anyon. Each of them corresponds to a unitary transformation on V_z^{mmmm} , which we denote by σ_1, σ_2 and σ_3 , respectively. Moreover, they satisfy the relation:

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_1\sigma_3 = \sigma_3\sigma_1.$$

This is just the relation that defines the braid group \mathcal{B}_4 on three generators. So we obtain a unitary representation of \mathcal{B}_4 on V_z^{mmmm} , which we denoted by $\rho(m, z)$. The computational power of the theory depends on the image of $\rho(m, z)$ in the special unitary group $SU(V_z^{mmmm})$.

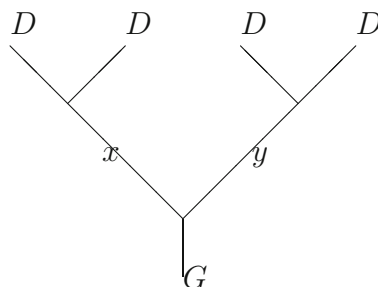
Definition 1 Given V_z^{mmmm} , we will call the unitary representation matrices $U(b)$ for braids b the braiding quantum circuits. The special braiding circuits $U(\sigma_i^\pm)$ for the braid generators σ_i^\pm will be called the braiding gates.

The same terminologies are extended to multi-anyons for multi-qudits.

Since our computational space is always a subspace of the braid group representation, the quantum circuits obtained from braiding quantum circuits are really their restrictions to the computational subspace. We will not make this distinction when no confusion will arise.

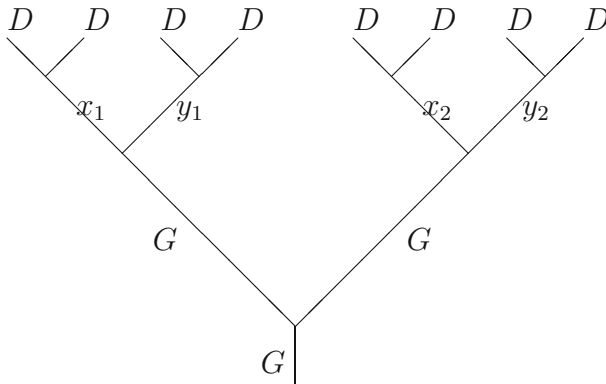
Now we specialize to $D(S_3)$. In order to compute the braiding matrices, we need all the $6j$ -symbols and R -symbols. For $D(S_3)$, all of them are listed in ‘‘Appendix 1.’’ To analyze the computational power of 1-qudit braiding circuits, we need all the representation matrices of \mathcal{B}_4 . We systematically analyzed all \mathcal{B}_4 representations in ‘‘Appendix 2.’’ These two important appendices are our technical advance.

The natural choice will be to encode a qudit in V_A^{DDDD} . Unfortunately, we did not succeed in finding a model that could be made universal even with measurements and ancillary states. Therefore, we turn to V_G^{DDDD} based on our knowledge of the braid group representations:



The space V_G^{DDDD} is nine-dimensional with a basis $\{|GG\rangle, |AG\rangle, |GA\rangle, |FC\rangle, |CF\rangle, |FH\rangle, |HF\rangle, |CH\rangle, |HC\rangle\}$. Let $U = span\{|GG\rangle, |AG\rangle, |GA\rangle\}$, $V = span\{\frac{1}{\sqrt{2}}(|FC\rangle + |CF\rangle), \frac{1}{\sqrt{2}}(|FH\rangle + |CH\rangle), \frac{1}{\sqrt{2}}(|HF\rangle + |HC\rangle)\}$ and $W = span\{\frac{1}{\sqrt{2}}(|FC\rangle - |CF\rangle), \frac{1}{\sqrt{2}}(|CH\rangle - |FH\rangle), \frac{1}{\sqrt{2}}(|HF\rangle - |HC\rangle)\}$. To remind ourselves that these bases are used as computational basis, we also write them as $\{|0\rangle_x, |1\rangle_x, |2\rangle_x\}$, $x = U, V, W$, where the subscript x indicates which subspace we are referring to, e.g., $|0\rangle_U = |GG\rangle$. The representation of \mathcal{B}_4 splits into the direct sum of a 6-dim irreducible summand $U \oplus V$ and a 3-dim irreducible summand W .

To encode 2-qutrits, we consider the following fusion tree:



The 2-qutrits are the tensor product of the two qutrits on the two branches. This encoding of 2-qutrits is called the *sparse encoding* because encoding with fewer anyons, called the *dense encoding*, is also possible. To encode n -qutrits, we simply use the tensor product of n such branches, so there are totally $4n$ anyons. See Fig. 1.

We will refer to the three qutrit models that encode 1-qutrit in the subspaces U , V and W , respectively, with the computational bases above as the *qutrit U -model*, *V -model* and *W -model*, respectively.

To analyze these models, we systematically investigate all relevant braid group representations in “Appendix 2.” Our results are summarized below in Tables 3 and 4.

Table 3 list of the dimensions and bases of V_z^{mmmm} , and Table 4 the basic properties of the representations $\rho(m, z)$. In Table 4, for each pair of anyons (m, z) , the corresponding row lists the dimension of the representation V_z^{mmmm} , the dimension of each irreducible subrepresentation in V_z^{mmmm} and the image of the braid group on each irreducible subrepresentation.

Note that the group of order 648 in the last row of Table 4 is isomorphic, as an abstract group, to $((\mathbb{Z}_3 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2)) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3 \rtimes \mathbb{Z}_2$. This isomorphism is given by the software package GAP.

Fig. 1 Encoding of n qutrits

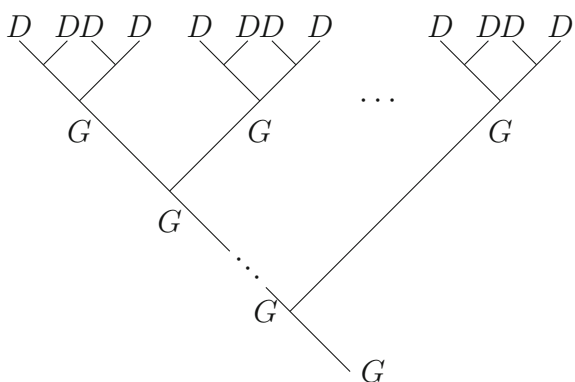


Table 3 Dimension and basis of V_z^{mmmm}

m, z	Dimension	Basis
C, A	3	$ AA\rangle, BB\rangle, CC\rangle$
C, B	3	$ CC\rangle, AB\rangle, BA\rangle$
C, C	5	$ CC\rangle, AC\rangle, CA\rangle, BC\rangle, CB\rangle$
D, A	5	$ AA\rangle, CC\rangle, FF\rangle, GG\rangle, HH\rangle$
D, B	4	$ CC\rangle, FF\rangle, GG\rangle, HH\rangle$
D, C	9	$ CC\rangle, AC\rangle, CA\rangle, GF\rangle, FG\rangle, GH\rangle, HG\rangle, FH\rangle, HF\rangle$
D, F	9	$ FF\rangle, AF\rangle, FA\rangle, GC\rangle, CG\rangle, GH\rangle, HG\rangle, CH\rangle, HC\rangle$
D, G	9	$ GG\rangle, AG\rangle, GA\rangle, FC\rangle, CF\rangle, FH\rangle, HF\rangle, CH\rangle, HC\rangle$
D, H	9	$ HH\rangle, AH\rangle, HA\rangle, GF\rangle, FG\rangle, GC\rangle, CG\rangle, FC\rangle, CF\rangle$
G, A	3	$ AA\rangle, BB\rangle, GG\rangle$
G, B	3	$ GG\rangle, AB\rangle, BA\rangle$
G, G	3	$ GG\rangle, AG\rangle, GA\rangle, BG\rangle, GB\rangle$

Table 4 Summary of the representations $\rho(m, z)$ on V_z^{mmmm}

m, z	Dimension of V_z^{mmmm}	Dimension of each irrep in V_z^{mmmm}		Image of $\rho(m, z)$ on each irrep	
C, A	3	2	1	$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$	1
C, B	3	3		S_4	
C, C	5	3	1	S_4	1
D, A	5	3	1	\mathcal{A}_4	1
D, B	4	2	2	$SL(2, \mathbb{F}_3)$	$SL(2, \mathbb{F}_3)$
D, C	Same as (D, F)				
D, F	9	8	1	$\sum(216)$	1
D, G	9	3	6	$\sum(216 * 3)$	$\sum(216 * 3)$
D, H	Same as (D, G)				
G, A	3	3		$D(9,1,1;2,1,1)$	
G, B	3	3		$D(18,1,1;2,1,1)$	
G, G	5	4	1	Group of order 648	1

2.4 Braiding, measurement and ancilla

Using the encoding above, we can simulate standard qutrit quantum circuits by braidings of D anyons. Concrete braiding quantum circuits are the braid group representation matrices with respect to the fusion tree basis. For 1-qutrit braiding circuits, we need to know the representation matrices of \mathcal{B}_4 —the 4-strand braid group, and for 2-qutrit braiding circuits, the representation matrices of \mathcal{B}_8 . Since both collections of matrices are finite [5], they are not sufficient to simulate the standard qutrit circuit model.

To gain extra computational power, we consider measurement and ancilla. In anyon theory, there are two kinds of measurements to determine the total charge of a collection of anyons: projective and interferometric. Both types of measurements always lead to some decoherence in the model. Therefore, ideally we should only use them at the end of the computation. Since we cannot avoid using them for WI anyons, we will allow ourselves to determine whether or not the total charge of two anyons is trivial in the middle of the computation. Then based on the outcome, we choose how to continue our computation. For this reason, we call such models *adaptive*.

Measurement 1 Let $\mathcal{M}_A = \{\Pi_A, \Pi_{A'}\}$ be the projective measurement onto the total charge= A sector and its complement. Then \mathcal{M}_A allows us to distinguish between the anyon A and other anyons, namely check whether a anyon is trivial or not. Moreover, the state after measurement for each outcome is still coherent.

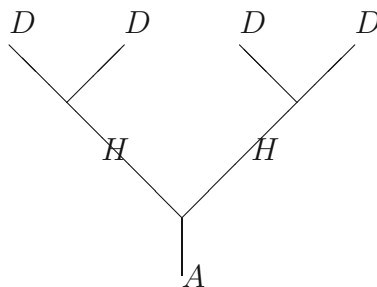
The next measurement that we use is problematic, but it is unavoidable due to our choices of computational subspaces. It allows us to project states back to the computational subspaces.

Measurement 2 Let S be a subspace of an anyonic space and S^\perp be its orthonormal complement. Then $\mathcal{M}_S = \{\Pi_S, \Pi_{S^\perp}\}$ is the projective measurement that projects a state to S or S^\perp .

For example, applying M_S to $S = U$ in V_G^{DDDD} , we obtain the orthogonal projection to $U = \text{span}\{|GG\rangle, |AG\rangle, |GA\rangle\}$ and its orthogonal complement $V \oplus W = \text{span}\{|FC\rangle, |CF\rangle, |FH\rangle, |HF\rangle, |CH\rangle, |HC\rangle\}$.

The main result of the paper is that braiding supplemented by measurements \mathcal{M}_A and \mathcal{M}_U leads to a universal gate set for the U -model and V -model. To make the qutrit W -model universal, we need to use the extra ancillary state:

Ancilla 1 The state in the following picture is denoted by $|H\rangle_A$.



Then our second result is that braiding supplemented by measurements \mathcal{M}_A and \mathcal{M}_U and ancillary state $|H\rangle_A$ leads to a universal gate set for the W -model.

2.5 A universal gate set for qutrits

Theoretically, there is no advantage to use qutrits instead of qubits. But there are anyon systems that are more natural to choose qutrits rather than qubits. This is the case when

we use WI anyons of quantum dimensions 3 or $\sqrt{3}$ for anyonic quantum computation. Moreover, there are some better numbers for qutrits distillation protocols which might provide some benefits for engineering [1, 8]. In this section, we prove that a particular convenient qutrit gate set for our purpose is universal for the standard qutrit circuit model.

The generalized Hadamard gate for qutrit is the following:

$$h = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

where $\omega = e^{\frac{2\pi i}{3}}$.

The SUM gate for qudits is a generalized version of *CNOT*, which maps basis element $|i, j\rangle$ to $|i, i + j \pmod 3\rangle$. To state our theorem, we need to define another measurement:

Measurement 3 Let $\mathcal{M}_{|0\rangle} = \{\Pi_{|0\rangle}, \Pi_{|0\rangle^\perp}\}$ be the projective measurement that is the orthogonal projection to $\text{span}\{|0\rangle\}$ and its orthogonal complement $\text{span}\{|1\rangle, |2\rangle\}$ in a qutrit.

Theorem 1 *The 1-qutrit classical gates, generalized Hadamard gate, SUM gate and Measurement 3 form a universal gate set for the standard qutrit quantum circuit model.*

2.5.1 Proof of Theorem 1

We fix $|0\rangle$ and $|1\rangle$ as a qubit and show that we can implement a universal set of qubit gates. More explicitly, we use the two-dimensional subspace $\mathbb{C}^2 = \text{span}\{|0\rangle, |1\rangle\}$ inside $\mathbb{C}^3 = \text{span}\{|0\rangle, |1\rangle, |2\rangle\}$ to do computations. During the computations, we will go out of the subspace \mathbb{C}^2 , and eventually come back to it. Though unnecessary, we can deduce universality for the qutrit models by encoding a qutrit with two qubits $\mathbb{C}^2 \otimes \mathbb{C}^2 \subset \mathbb{C}^3 \otimes \mathbb{C}^3$. That is, we use $|00\rangle, |01\rangle, |10\rangle$ to encode $|0\rangle, |1\rangle, |2\rangle$, respectively. And the basis element $|11\rangle$ is left unused.

Our strategy of proof follows from that of [9], and some of the lemmas below are stated in [9] as exercises.

Note that with 1-qutrit classical gates, the generalized Hadamard gate h and Measurement 3, we can easily construct the following ancilla and measurements:

1. $|i\rangle, i = 0, 1, 2$.
2. $|\widetilde{i}\rangle = \sum_{j=0}^2 \omega^{ij} |j\rangle = h|i\rangle, i = 0, 1, 2$.
3. Projection of a 1-qutrit state to any computational state, preserving the coherence of the orthogonal complement. For example, projection to $\text{span}\{|0\rangle, |1\rangle\}$ and its complement $\text{span}\{|2\rangle\}$.
4. Measurement of a qutrit in the standard computational basis.
5. Projection to $\text{span}\{|\widetilde{1}\rangle, |\widetilde{2}\rangle\}$ and its complement $\text{span}\{|\widetilde{0}\rangle\}$
6. Measurement of a qubit in the standard basis if we take $\{|0\rangle, |1\rangle\}$ as the computational basis. This follows from 4).

From the set of operations given in Theorem 1, we show that we can construct the qutrit (qubit) gates (measurements) in Lemmas 1, 2 and 3.

We define the qutrit gate FLIP₂ by the map: FLIP₂|0⟩ = |0⟩, FLIP₂|1⟩ = |1⟩, FLIP₂|2⟩ = -|2⟩.

Lemma 1 *The gate FLIP₂, can be constructed.*

Proof To obtain FLIP₂, we first construct the ancilla |ψ⟩ = $\frac{1}{\sqrt{3}}(|0\rangle - |1\rangle + |2\rangle)$ as follows.

Prepare the state $\widetilde{|1\rangle|2\rangle}$ and project each qutrit to the space $span\{|0\rangle, |1\rangle\}$ to obtain the state $|\eta\rangle = \frac{1}{2}(|0\rangle + \omega|1\rangle) \otimes (|0\rangle + \omega^2|1\rangle)$. Apply the SUM gate to $|\eta\rangle$ and then project the first qutrit of the resulting state to the space $span\{\widetilde{|0}\rangle\}$. It is easy to see on the second qutrit we get the state |ψ⟩.

Now for a state $|\phi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle$, apply the SUM gate to $|\phi\rangle|\psi\rangle$ and then measure the second qutrit in the standard basis. If the outcome is |0⟩, then the first qutrit is $c_0|0\rangle + c_1|1\rangle - c_2|2\rangle$. If the outcome is |1⟩, then the first qutrit is $-c_0|0\rangle + c_1|1\rangle + c_2|2\rangle$, and if the outcome is |2⟩, then the first qutrit is $c_0|0\rangle - c_1|1\rangle + c_2|2\rangle$. Moreover, the probability for each case is $\frac{1}{3}$. Therefore, this process changes the sign of some coefficient randomly. By repeating this process, we will get the gate FLIP₂. □

Lemma 2 *The 3-qubit gate $\bigwedge^2(\sigma_z)$ which maps $|i, j, k\rangle$ to $(-1)^{ijk}|i, j, k\rangle$ can be constructed. In particular, $\bigwedge(\sigma_z)$ and σ_z can be constructed since we have the ancilla |1⟩.*

Proof Combining the gate FLIP₂ obtained in Lemma 1 and the SUM gate, one can construct the following 2-qutrit and 3-qutrit gates.

$$|i, j\rangle \mapsto \begin{cases} -|i, j\rangle & i + j = 2 \pmod 3 \\ |i, j\rangle & \text{else} \end{cases} \tag{2.1}$$

$$|i, j, k\rangle \mapsto \begin{cases} -|i, j, k\rangle & i + j + k = 2 \pmod 3 \\ |i, j, k\rangle & \text{else} \end{cases} \tag{2.2}$$

When applying them to the state $|i, j\rangle$ (or $|i, j, k\rangle$), we can describe the above two gates as “flip the sign if $i + j = 2 \pmod 3$ ”(or “flip the sign if $i + j + k = 2 \pmod 3$ ”).

One can check applying the following four gates to a 3-qubit state $|i, j, k\rangle$ successively gives rise to $\bigwedge^2(\sigma_z)$.

- “flip the sign if $i + j + k = 2 \pmod 3$,”
- “flip the sign if $i + j = 2 \pmod 3$,”
- “flip the sign if $i + k = 2 \pmod 3$,”
- “flip the sign if $j + k = 2 \pmod 3$,”

Note that here i, j, k are either 0 or 1. □

Let $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, which are the eigenstates of the qubit gate σ_x . Note that the state $|+\rangle$ can be obtained by projecting the state $\widetilde{|0}\rangle$ to the space $span\{|0\rangle, |1\rangle\}$.

Lemma 3 *Measurement of σ_x can be constructed on a qubit.*

Proof For an arbitrary 1-qubit state $\alpha|+\rangle + \beta|-\rangle$, a measurement of σ_x would result in the state $|+\rangle$ with probability $|\alpha|^2$, and in the state $|-\rangle$ with probability $|\beta|^2$.

We denote the measurement which projects a state to $span\{|\widetilde{0}\rangle\}$ and its complement $span\{|\widetilde{1}\rangle, |\widetilde{2}\rangle\}$ by \mathcal{M}_1 and denote the measurement which projects to $span\{|0\rangle, |1\rangle\}$ and its complement $span\{|2\rangle\}$ by \mathcal{M}_2 .

Note that $|-\rangle$ is orthogonal to $|\widetilde{0}\rangle$ while $|+\rangle$ is not. So if a state results in $|\widetilde{0}\rangle$ after \mathcal{M}_1 , then the corresponding probability only depends on the $|+\rangle$ component. We explain this idea explicitly below to construct the measurement of σ_x .

Consider the following procedure.

$$\begin{aligned}
 \mathcal{O} : \alpha|+\rangle + \beta|-\rangle &\xrightarrow{\mathcal{M}_1} \begin{matrix} \text{output} \xrightarrow{Pr=\frac{2|\alpha|^2}{3}} |\widetilde{0}\rangle \\ Pr=1-\frac{2|\alpha|^2}{3} \end{matrix} \longrightarrow |+\rangle \\
 &\swarrow \\
 \frac{\alpha|+\rangle + \beta|-\rangle - \frac{2\alpha}{\sqrt{6}}|\widetilde{0}\rangle}{\sqrt{1-\frac{2|\alpha|^2}{3}}} &\xrightarrow{\mathcal{M}_2} \begin{matrix} \text{output} \xrightarrow{Pr=\frac{2|\alpha|^2}{9(1-\frac{2|\alpha|^2}{3})}} |2\rangle \\ Pr=\frac{9-8|\alpha|^2}{9-6|\alpha|^2} \end{matrix} \longrightarrow |+\rangle \\
 &\swarrow \\
 \frac{\alpha|+\rangle + 3\beta|-\rangle}{\sqrt{9-8|\alpha|^2}} &
 \end{aligned} \tag{2.3}$$

So the procedure \mathcal{O} consists of two measurements \mathcal{M}_1 and \mathcal{M}_2 . If the outcome is $|\widetilde{0}\rangle$ after \mathcal{M}_1 or is $|2\rangle$ after \mathcal{M}_2 , then we prepare the state $|+\rangle$, namely we take the appearance of these two cases as the outcome $|+\rangle$. The probability for either of these two cases to happen is $\frac{8|\alpha|^2}{9}$. Otherwise, we get the state $\frac{\alpha|+\rangle + 3\beta|-\rangle}{\sqrt{9-8|\alpha|^2}}$ with probability $1 - \frac{8|\alpha|^2}{9}$, and then, we iterate the procedure \mathcal{O} until the above two cases happen or the required accuracy is satisfied.

More explicitly, let the resulting state be $|\psi\rangle_n = \alpha_n|+\rangle + \beta_n|-\rangle$ after iterating the procedure \mathcal{O} n times with no $|+\rangle$ outcome and let b_n be the probability to obtain $|\psi\rangle_n$ from $|\psi\rangle_{n-1}$ via the n -th procedure. Then we have the following equations.

$$\alpha_n = \frac{\alpha_{n-1}}{\sqrt{9-8|\alpha_{n-1}|^2}}, \quad \beta_n = \frac{3\beta_{n-1}}{\sqrt{9-8|\alpha_{n-1}|^2}}, \quad b_n = 1 - \frac{8|\alpha_{n-1}|^2}{9} = \frac{|\alpha_{n-1}|^2}{9|\alpha_n|^2} \tag{2.4}$$

From the above equations, we have

$$|\alpha_n|^2 = \frac{|\alpha|^2}{(1-|\alpha|^2)9^n + |\alpha|^2}, \quad \prod_{i=1}^n b_i = |\beta|^2 + \frac{|\alpha|^2}{9^n} \tag{2.5}$$

So the probability for iterating the procedure n times with no $|+\rangle$ outcome is $b = \prod_{i=1}^n b_i = |\beta|^2 + \frac{|\alpha|^2}{9^n}$, which is very close to $|\beta|^2$ when n is large. Moreover,

$|\langle +|\psi_n\rangle|^2 = |\alpha_n|^2$ which is close to zero, namely ψ_n is almost equal to $|-\rangle$ up to a phase. Therefore, it is reasonable to treat the case that no $|+\rangle$ appears within n procedures for some proper large n , as the outcome $|-\rangle$.

To sum up, after iterating the procedure n times, we can get the state $|+\rangle$ with probability $1 - b = (1 - \frac{1}{9^n})|\alpha|^2$ and $|-\rangle$ with probability $b = |\beta|^2 + \frac{|\alpha|^2}{9^n}$. If we take n large enough, we get the measurement of σ_x with required accuracy.

Lemma 4 [13] *The following set of qubit operations are universal for quantum computation:*

1. Create the state $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, $|0\rangle$ and $|1\rangle$
2. Measure σ_z .
3. Measure σ_x .
4. The Toffoli gate $T = \bigwedge^2(\sigma_x)$.

For a proof of this lemma, see [13].

Lemma 5 [9] *The following set of qubit operations are universal for quantum computation:*

1. Create the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
2. Measure σ_z .
3. Measure σ_x .
4. The gate $\bigwedge^2(\sigma_z)$.

Proof We prove this lemma by showing that the set of operations here can be used to implement all the operations in Lemma 4.

Since we can measure σ_x and σ_z , it is clear that $|-\rangle$, $|0\rangle$ and $|1\rangle$ all can be created from $|+\rangle$. Thus, it suffices to show that the Toffoli gate can be created.

For notational convenience, we also denote $|\tilde{0}\rangle = |+\rangle$, $|\tilde{1}\rangle = |-\rangle$ in the following proof. The readers should not be confused with this notation and the one we used for a qutrit, since for the moment we only work in qubit space.

With the ancilla $|1\rangle$ and the gate $\bigwedge^2(\sigma_z)$, we can get the gates $\bigwedge(\sigma_z)$ and σ_z .

Next we do the following procedure which creates a “gate” H or $\sigma_x H$,

$$|i\rangle|\tilde{0}\rangle \xrightarrow{\bigwedge(\sigma_z)} |i\rangle|\tilde{i}\rangle \xrightarrow{\text{Measure}(\sigma_x)_1} \begin{cases} |\tilde{0}\rangle|\tilde{i}\rangle & \text{outcome is } 1 \\ |\tilde{1}\rangle(-1)^i|\tilde{i}\rangle & \text{outcome is } -1 \end{cases}$$

By measuring $(\sigma_x)_1$, we mean measuring σ_x on the first qubit.

One checks that both probabilities are $\frac{1}{2}$. If the outcome is -1 , then we continue to apply the gate σ_z on the first qubit so that the state becomes $|\tilde{0}\rangle(-1)^i|\tilde{i}\rangle$.

Notice that our ancilla starts from the second qubit while ends on the first qubit, *i.e.*, the working qubit and ancilla qubit are switched. But we show below that this is not a problem.

Therefore, if the outcome is 1, we produced the gate qubit Hadamard gate H , and otherwise we produced $\sigma_x H$. We name this sequence of operations by \mathcal{A} .

Now we produce the gate T .

$$\begin{aligned}
 |i, j, k\rangle|\tilde{0}\rangle &\xrightarrow{\mathcal{A}_{3,4}} \begin{cases} |i, j, \tilde{0}, \tilde{k}\rangle & \text{outcome } 1 \\ (-1)^k|i, j, \tilde{0}, \tilde{k}\rangle & \text{outcome } -1 \end{cases} \xrightarrow{\wedge^2(\sigma_z)_{1,2,4}} \\
 \begin{cases} |i, j, \tilde{0}, \widetilde{ij+k}\rangle \\ (-1)^k|i, j, \tilde{0}, \widetilde{ij+k}\rangle \end{cases} &\xrightarrow{\mathcal{A}_{4,3}} \begin{cases} |i, j, ij+k, \tilde{0}\rangle & \text{outcome } (1, 1) \\ (-1)^k|i, j, ij+k, \tilde{0}\rangle & \text{outcome } (-1, 1) \\ |i, j, ij+k+1, \tilde{0}\rangle & \text{outcome } (1, -1) \\ (-1)^k|i, j, ij+k+1, \tilde{0}\rangle & \text{outcome } (-1, -1) \end{cases}
 \end{aligned}$$

In the diagram above, $\mathcal{A}_{3,4}$ means applying the operation \mathcal{A} with the third qubit as working bit and the fourth qubit as ancilla. Each pair of outcome happens with probability $\frac{1}{4}$.

If the outcome is (1,1) or (1,-1), we do nothing.

If the outcome is (-1,1) or (-1,-1), we apply the gate $(\sigma_z)_3 \wedge (\sigma_z)_{1,2}$ to get the state $|i, j, ij+k\rangle$ or $-|i, j, ij+k+1, \tilde{0}\rangle$. The overall phase is not important.

Therefore, if the outcome is (1,1) or (-1,1), we produced the gate T . Otherwise we got the gate $(\sigma_x)_3 T$. Both probabilities are $\frac{1}{2}$.

In the latter case, we repeat the procedure, then we either go back to the original state with probability $\frac{1}{2}$, or we go to the state $|i, j, k+1\rangle$ also with probability $\frac{1}{2}$. Repeat the procedure again. It is easy to see that after doing this procedure at most 3 times, the probability to get the state $|i, j, ij+k\rangle$ is $\frac{1}{2} + \frac{1}{2^2}$.

After at most $2n - 1$ times, the probability to get T is $\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$.

Therefore, after repeating enough times, we will eventually produce the gate T . \square

By the lemmas above in this subsection, all the operations in Lemma 5 can be created from the operations given in Theorem 1 if we pick a qubit from the qutrit space. Thus, Lemma 5 implies Theorem 1.

3 Universal adaptive anyonic computing models

In this section, we prove that the U -model, V -model and W -model in Sect. 2.3 can be made universal provided measurement and ancilla are allowed besides braiding.

Recall that $U = span\{|GG\rangle, |AG\rangle, |GA\rangle\}$, $V = span\{\frac{1}{\sqrt{2}}(|FC\rangle + |CF\rangle), \frac{1}{\sqrt{2}}(|FH\rangle + |CH\rangle), \frac{1}{\sqrt{2}}(|HF\rangle + |HC\rangle)\}$ and $W = span\{\frac{1}{\sqrt{2}}(|FC\rangle - |CF\rangle), \frac{1}{\sqrt{2}}(|CH\rangle - |FH\rangle), \frac{1}{\sqrt{2}}(|HF\rangle - |HC\rangle)\}$. The computational basis for the three models are denoted as $\{|0\rangle_x, |1\rangle_x, |2\rangle_x\}$ corresponding to the pair of anyons above, $x = U, V, W$, where the subscript x indicates which subspace we are referring to.

Our main theorems are as follows:

Theorem 2 *Braiding quantum gates and Measurements 1 and 2 provide a universal gate set for the qutrit U -model and V -model.*

Theorem 3 *A universal gate set for the W -model can be constructed from braidings and Measurements 1, 2 when the ancillary state 1 is used.*

The proof of Theorem 2 is given in the next subsection and the proof of Theorem 3 in Sect. 3.2.

3.1 Universality for U - and V -models

$U \oplus V$ is a 6-dim irreducible representation of \mathcal{B}_4 . Under the basis $span\{|0\rangle_U, |1\rangle_U, |2\rangle_U, |0\rangle_V, |1\rangle_V, |2\rangle_V\}$, the generators σ_i 's have the following matrices:

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} \omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \end{pmatrix} \\ \sigma_2 &= \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega & \sqrt{2}\omega^2 & \sqrt{2} & \sqrt{2} \\ \omega & 1 & \omega & \sqrt{2} & \sqrt{2} & \sqrt{2}\omega^2 \\ \omega & \omega & 1 & \sqrt{2} & \sqrt{2}\omega^2 & \sqrt{2} \\ \sqrt{2}\omega^2 & \sqrt{2} & \sqrt{2} & -\omega & -\omega^2 & -\omega^2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2}\omega^2 & -\omega^2 & -\omega & -\omega^2 \\ \sqrt{2} & \sqrt{2}\omega^2 & \sqrt{2} & -\omega^2 & -\omega^2 & -\omega \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} \omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Let $p = \sigma_1\sigma_2\sigma_1$ and $q = \sigma_3\sigma_2\sigma_3$. Then

$$\begin{aligned} p^2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ q^2 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\ p^2q^2p^2 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

to construct γ within n times is $1 - (\frac{5}{9})^n$. Therefore, one can obtain γ exponentially fast.

Similarly, one can construct γ^{-1} . □

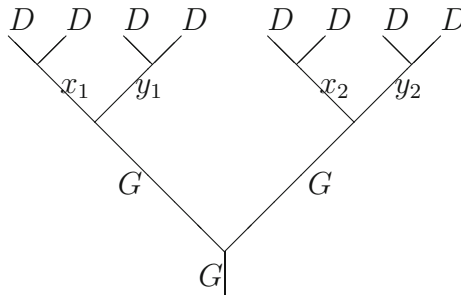
The following lemma shows that Measurement 3 can be constructed in both U and V .

Lemma 7 *Using Measurements 1, 2 and braiding, one can perform Measurement 3 in both the spaces U and V .*

Proof Note that we used the notation $|0\rangle_U = |GG\rangle$, $|1\rangle_U = |AG\rangle$, $|2\rangle_U = |GA\rangle$. Given a state $|\psi\rangle = a|GG\rangle + b|AG\rangle + c|GA\rangle$ in U , we apply Measurement 1 to the left half of the state, i.e, we check whether or not the first pair of D anyons in the 1-qudit splitting tree has total trivial charge. This is essentially the projection to $span\{|AG\rangle\}$ and its orthogonal complement in U , namely the projection to $span\{|1\rangle_U\}$ and $span\{|0\rangle_U, |2\rangle_U\}$. Since we have all the 1-qudit classical gates on U , it is clear that Measurement 3 in U can be constructed.

Measurement 3 in V follows from Lemma 6 that one can construct the transformation γ , γ^{-1} to go back and forth between U and V . □

Up to now, we only considered gates and operations on one qudit. Next, we want to construct a 2-qudit gate, the Controlled-Z gate $\bigwedge(Z)$ which maps $|i, j\rangle$ to $\omega^{ij}|i, j\rangle$.



We use the above fusion tree to encode 2-qudits. Let $s_1 = \sigma_2\sigma_1\sigma_3\sigma_2$ namely, s_1 is the braiding of the first pair with the second pair. Similarly let $s_2 = \sigma_4\sigma_3\sigma_5\sigma_4$, $s_3 = \sigma_6\sigma_5\sigma_7\sigma_6$. Clearly s_1 exchanges x_1 with y_1 with a phase in the above 2-qudit splitting tree, namely it maps $|x_1, y_1; x_2, y_2\rangle$ to $|y_1, x_1; x_2, y_2\rangle$ up to a phase. Similarly, s_3 exchanges x_2 with y_2 . The gate s_2 is much more complicated since it involves F -moves. Let $CrIZ = s_1^{-1}s_2^2s_1s_3^{-1}s_2^2s_3$. Through direct calculations, we found that $CrIZ$ is a diagonal matrix. Moreover, when restricted to the space U , $CrIZ$ is exactly the Controlled-Z gate $\bigwedge(Z)$. Again, via the transformation γ , one also obtains the Controlled-Z gate in the space V .

The SUM gate maps $|i, j\rangle$ to $|i, i + j\rangle$ and can be obtained by conjugating $\bigwedge(Z)$ via the Hadamard. Explicitly,

$$\text{SUM} = (Id \otimes h) \bigwedge(Z)^{-1} (Id \otimes h^{-1}).$$

So we can also construct the SUM gate in the space U and V .

To sum up, with Measurements 1, 2 and braiding, we can construct all the 1-qutrit classical gates, generalized Hadamard gate, SUM gate and Measurement 3 in both the spaces U and V .

Finally, Theorem 2 follows from Theorem 1 and the arguments in this subsection.

3.2 Universality for W -model

In this subsection, we examine the representation on W . Under the basis of W given by $\{|0\rangle_W, |1\rangle_W, |2\rangle_W\}$, the σ_i 's have the matrices:

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix} & \sigma_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}i}{6} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} \\ -\frac{1}{2} + \frac{\sqrt{3}i}{6} & \frac{1}{2} + \frac{\sqrt{3}i}{6} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} \\ -\frac{1}{2} + \frac{\sqrt{3}i}{6} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} & \frac{1}{2} + \frac{\sqrt{3}i}{6} \end{pmatrix} \end{aligned}$$

The same as last subsection, define $p = \sigma_1\sigma_2\sigma_1$, $q = \sigma_3\sigma_2\sigma_3$. Then

$$p^2 = -\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad q^2 = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

So p^2 and q^2 generate all the 1-qutrit classical gates in W .

Also from σ_1 and σ_3 , we obtain the generalized Z -gate and Phase gate P :

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

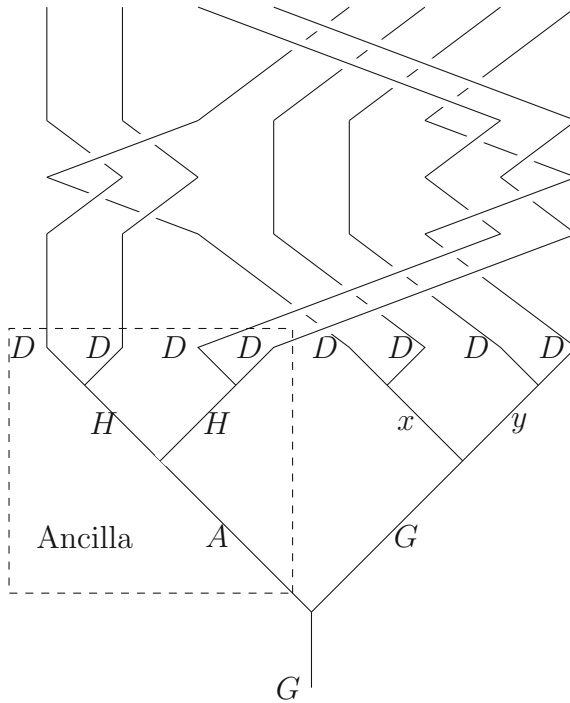
where Z maps $|i\rangle$ to $\omega^i|i\rangle$ and P maps $|i\rangle$ to $\omega^{\frac{i^2-i}{2}}|i\rangle$.

Moreover, let $h' = q^2pq^2$, then $h' = \frac{1}{\sqrt{3}i} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$, which is exactly the generalized Hadamard gate up to a phase.

Therefore, in the space W , we obtained the classical 1-qutrit gates, generalized Z -gate, the Phase gate and the generalized Hadamard gate by braiding.

Now we turn to constructing the 2-qutrit gate $\wedge(Z)$. One may try the same braiding method as we did for the space U . But it turns out that braiding does not work for W . Instead, we try to construct a transformation similar to γ .

Consider the following picture of braiding.



Let $P = \sigma_6\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4$, $Q = \sigma_2\sigma_1\sigma_1\sigma_2\sigma_6\sigma_7\sigma_7\sigma_6$, and let $R = P^{-1}QP$. Then the braiding in the picture is given by R .

We denote the state in the picture before braiding by $|H\rangle_A |xy\rangle$. Then the braiding R gives the following transformation:

$$|H\rangle_A |i\rangle_W \mapsto \frac{1}{2}(-|H\rangle_A |i\rangle_W + |H\rangle_B |i\rangle_V - \sqrt{2}|H\rangle_B | -i\rangle_U)$$

and

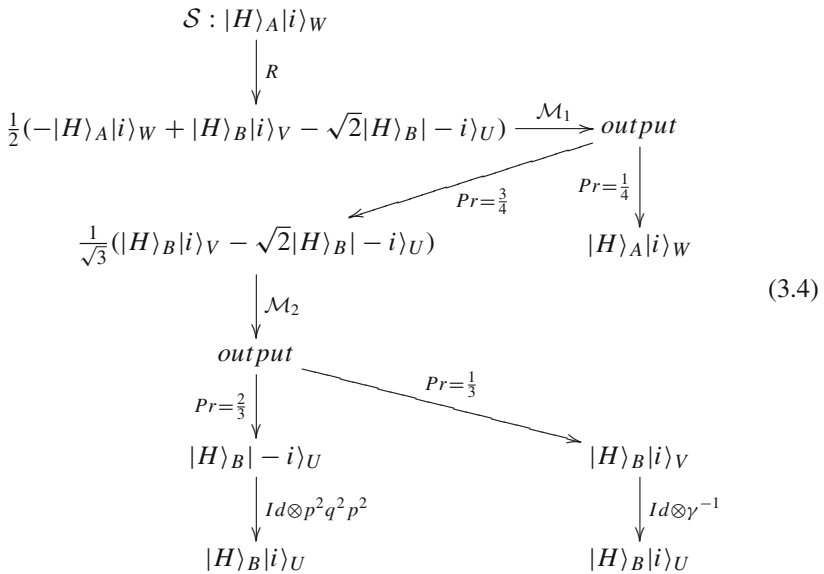
$$|H\rangle_B |i\rangle_U \mapsto \frac{1}{\sqrt{2}}(|H\rangle_A | -i\rangle_W + |H\rangle_B | -i\rangle_V)$$

where $i = 0, 1, 2$ and $-i$ is taken to be modulo 3.

Define a unitary transformation $\beta : |H\rangle_A \otimes W \rightarrow |H\rangle_B \otimes U$, $\beta(|H\rangle_A |i\rangle_W) = |H\rangle_B |i\rangle_U$. Here $|H\rangle_A$ is the ancilla.

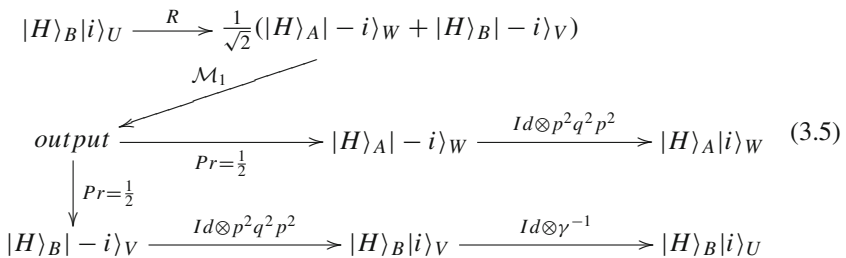
Lemma 8 *With braiding, Measurements 1, 2 and Ancilla 1, the transformation β and β^{-1} can be constructed with probability approaching to 1 exponentially fast in the number of measurements and the gates applied.*

Proof In the following diagram, \mathcal{M}_1 means applying Measurement 1 to the first qudit (the ancilla part) to check whether the total charge is trivial or not and \mathcal{M}_2 is Measurement 2 applied to the second qudit. Consider the following procedure \mathcal{S} :



Starting from the state $|H\rangle_A |i\rangle_W$ with $|H\rangle_A$ as ancilla, we apply the procedure \mathcal{S} to it. From the diagram above, one can see that there is a probability of $\frac{1}{4}$ for the state to remain unchanged in which case we would apply the procedure again. Otherwise, the state is transformed to $|H\rangle_B |i\rangle_U$, namely the transformation β is constructed. Note that in the above procedure, the states in each outcome are considered up to a global phase, which is of course irrelevant. By repeating the procedure \mathcal{S} , β can be obtained exponentially fast.

β^{-1} can be constructed in a similar way by repeated use of the following procedure \mathcal{T} :



□

By going back and forth between W and U via β and β^{-1} , any operation in the space U can be performed in W accordingly. In particular, the Controlled-Z gate and Measurement 3 can be constructed in W .

Collecting the results in this subsection, we finish the proof of Theorem 3.

Appendix 1: Solutions of the $D(\mathcal{S}_3)$ fusion rules

Given a set of fusion rules, it is highly non-trivial to solve for all $6j$ symbols even with software packages. Though $D(\mathcal{S}_3)$ is a large anyon system, recent progress makes it possible to solve for all modular categories with the same fusion rules. In the following, we list the complete data only for the $D(\mathcal{S}_3)$. See Sect. 2.2 for an explanation of the notations that we use for F -matrices and R -matrices.

The subcategory spanned by $\{A, B, G\}$ is a near-group category of type $(\mathbb{Z}_2, 1)$ and analyzed completely in [20]. (The objects A, B, G here are the ϵ, g, m in [20], respectively.)

The only monoidal structure which allows braiding is the following one [20]. When we list $6j$ symbols, all the admissible ones that are equal to 1 are omitted.

$$F_{GGG}^{BGG} = F_{GGG}^{GBG} = F_{GGG}^{GGB} = F_{BGG}^{GGG} = -1,$$

$$F_G^{GGG} = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

Note that we normalize the trivalent basis to obtain unitary F matrices, while the original F matrices in [20] are not unitary.

There are three braiding structures on the subcategory depending on a choice of $\omega \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$.

$$R_A^{AA} = R_B^{AB} = R_A^{BA} = R_A^{BB} = R_G^{AG} = R_G^{GA} = 1,$$

$$R_G^{BG} = R_G^{GB} = -1,$$

$$R_A^{GG} = \omega^2,$$

$$R_B^{GG} = -\omega^2,$$

$$R_G^{GG} = \omega.$$

The subcategory is balanced for all choices of $\omega \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. Siehler claimed that the $\omega = 1$ structure is balanced but $\omega \neq 1$ structures are not (without explicit proof, see Proposition 10.1 in [20]). However, it is easy to see that all structures are balanced. Balancing equation is

$$Bal_c^{a,b} : \theta_c = \theta_a \theta_b R_c^{a,b} R_c^{b,a}$$

With twists $\theta_A = 1, \theta_B = 1, \theta_G = \omega$, all non-trivial balancing equations $Bal_G^{B,G}, Bal_G^{G,B}, Bal_B^{G,G}, Bal_G^{G,G}$ hold for any third root of unity ω .

However, the case $\omega = 1$ cannot be extended to get a braiding of the whole category. Here is the argument. Consider the hexagon equation on G summand in the product $D \otimes D \otimes G$ with the intermediate summand D in $G \otimes D$ and summand E in $D \otimes G$, which is $R_E^{DG} F_{G;ED}^{DGD} R_D^{DG} = F_{G;EA}^{DDG} R_G^{AG} F_{G;AD}^{GDD} + F_{G;EG}^{DDG} R_G^{GG} F_{G;GD}^{GDD}$. With the value

of $R_G^{GG} = 1$, the right-hand-side is $\frac{1}{\sqrt{2}} \cdot 1 \cdot \frac{1}{\sqrt{2}} + (-\frac{1}{\sqrt{2}}) \cdot 1 \cdot \frac{1}{\sqrt{2}} = 0$, while $F_{G;ED}^{DGD} = -\frac{\sqrt{3}}{2}$, thus one of the two R -symbols on the left-hand side must be zero, a contradiction.

The other two cases with $\omega = e^{2\pi i/3}, e^{4\pi i/3}$ can both be extended to the whole category to get a braiding. Moreover, their braiding structures are conjugate to each other; thus, we assume that $\omega = e^{2\pi i/3}$ from now on.

There are three monoidal structures that extend to all other simple objects. We will focus on the structure that we used in this paper. For simplicity, let $\mathcal{G} = \{A, B\}$, $\mathcal{C}_1 = \{G\}$, $\mathcal{C}_2 = \{D, E\}$, and $\mathcal{C}_3 = \{C, F, H\}$. In the following, we list associativity matrices according to types upon three upper objects in F_d^{abc} . For example, \mathcal{GGC}_2 -type contains all associativity matrices with two objects from \mathcal{G} and one from \mathcal{C}_2 . Using this notation, all \mathcal{GGG} -, \mathcal{GGC}_1 -, $\mathcal{GC}_1\mathcal{C}_1$ -, and $\mathcal{C}_1\mathcal{C}_1\mathcal{C}_1$ -types are given above.

A. 1 The rest of the 6j symbols

Beside the 6j-symbols above, the rest are as follows:

\mathcal{GGC}_2 -type:

- $F_D^{BDB} = F_E^{BEB} = -1$

$\mathcal{GC}_1\mathcal{C}_2$ -type:

- -1 for $F_E^{BDG}, F_D^{BEG}, F_E^{GDB}, F_D^{GEB}$

$\mathcal{GC}_1\mathcal{C}_3$ -type:

- -1 for $F_H^{BGC}, F_F^{BGH}, F_C^{BGH}, F_C^{BFG}, F_F^{BCG}, F_H^{BCG}, F_C^{BHG}, F_H^{GBF}, F_H^{GBC}, F_C^{GBH}, F_C^{GFB}, F_F^{GCB}, F_H^{GCB}, F_C^{GHB}, F_H^{FBG}, F_H^{FCBG}, F_H^{FCGB}, F_C^{HGB}, F_F^{HGB}, F_C^{HGB}$

$\mathcal{GC}_2\mathcal{C}_2$ -type:

- -1 for $F_F^{BDD}, F_F^{BED}, F_B^{DBD}, F_G^{DBE}, F_C^{DBE}, F_H^{DBE}, F_F^{DDB}, F_F^{DEB}, F_G^{EBD}, F_C^{EBD}, F_H^{EBD}, F_B^{EBE}$

$\mathcal{GC}_2\mathcal{C}_3$ -type:

- -1 for $F_E^{BDC}, F_E^{BDH}, F_D^{BEC}, F_D^{BEH}, F_D^{BFD}, F_E^{BFD}, F_D^{DBF}, F_D^{DFB}, F_E^{DFB}, F_D^{EBF}, F_D^{FBD}, F_D^{FBE}, F_E^{CDB}, F_D^{CEB}, F_E^{HDB}, F_D^{HEB}$

$\mathcal{GC}_3\mathcal{C}_3$ -type:

- -1 for $F_F^{BFF}, F_H^{BFC}, F_G^{BFH}, F_C^{BCC}, F_G^{BCH}, F_G^{BHC}, F_F^{BHC}, F_H^{BHH}, F_F^{FBF}, F_G^{FBC}, F_C^{FBH}, F_F^{FFB}, F_G^{CBF}, F_C^{CBC}, F_G^{CBH}, F_H^{CFB}, F_C^{CCB}, F_G^{CHB}, F_F^{CHB}, F_C^{HBF}, F_G^{HBC}, F_H^{HHH}, F_G^{HFB}, F_G^{HCB}, F_H^{HHB}$

$\mathcal{C}_1\mathcal{C}_1\mathcal{C}_2$ -type:

- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for $F_D^{G.G.D}, F_E^{GGE}, F_D^{DGG}, F_E^{EGG}$

- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ for $F_E^{G,G,D}, F_D^{GGE}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ for F_E^{DGG}, F_D^{EGG}
- $\frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$ for F_D^{GDG}
- $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$ for F_E^{GDG}, F_D^{GEG}
- $\frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$ for F_E^{GEG}

$\mathcal{C}_1\mathcal{C}_1\mathcal{C}_3$ - type:

- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for $F_F^{GGF}, F_C^{GGC}, F_H^{GGH}, F_F^{FGG}, F_C^{CGG}, F_H^{HGG}$
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for $F_F^{GFG}, F_C^{GCG}, F_H^{GHG}$

$\mathcal{C}_1\mathcal{C}_2\mathcal{C}_2$ - type:

- -1 for F_B^{DGE}, F_B^{EGD}
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for $F_G^{GDD}, F_C^{GDD}, F_H^{GDD}, F_G^{GEE}, F_C^{GEE}, F_H^{GEE}, F_G^{DDG}, F_C^{DDG}, F_H^{DDG}, F_G^{EEG}, F_C^{EEG}, F_H^{EEG}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ for $F_F^{GDD}, F_F^{GED}, F_G^{DEG}, F_H^{DEG}, F_G^{EDG}, F_F^{EDG}, F_C^{EDG}, F_F^{EEG}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ for $F_G^{GDE}, F_F^{GDE}, F_C^{GDE}, F_G^{GED}, F_H^{GED}, F_F^{GEE}, F_F^{DDG}, F_F^{DEG}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$ for $F_H^{GDE}, F_C^{GED}, F_C^{DEG}, F_H^{EDG}$
- $\frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$ for F_G^{DGD}, F_F^{DGD}
- $\frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$ for F_C^{DGD}
- $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for F_H^{DGD}
- $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$ for F_G^{DGE}, F_G^{EGD}
- $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & -1 \\ 1 & -\sqrt{3} \end{pmatrix}$ for F_F^{DGE}
- $\frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}$ for F_C^{DGE}, F_C^{EGD}

- $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ for F_H^{DGE}, F_H^{EGD}
- $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{pmatrix}$ for F_F^{EGD}
- $\frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$ for F_G^{EGE}
- $\frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$ for F_F^{EGE}, F_C^{EGE}
- $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ for F_H^{EGE}

$\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3$ - type:

- $\frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$ for F_D^{GDF}, F_D^{FDG}
- $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & -1 \\ 1 & -\sqrt{3} \end{pmatrix}$ for F_E^{GDF}, F_D^{FEG}
- $\frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$ for F_D^{GDC}, F_D^{CDG}
- $\frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}$ for $F_E^{GDC}, F_D^{GEC}, F_E^{CDG}, F_D^{CEG}$
- $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for F_D^{GDH}, F_D^{HDG}
- $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ for $F_E^{GDH}, F_D^{GEH}, F_E^{HDG}, F_D^{HEG}$
- $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{pmatrix}$ for F_D^{GEF}, F_E^{FDG}
- $\frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$ for $F_E^{GEF}, F_E^{GEC}, F_E^{FEG}, F_E^{CEG}$
- $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ for F_E^{GEH}, F_E^{HEG}
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ for $F_D^{GFD}, F_E^{GFD}, F_E^{DGF}, F_E^{DGC}, F_E^{DHG}, F_E^{EGF}, F_D^{EGH}, F_D^{EFG}, F_E^{EFG}, F_D^{ECG}, F_D^{FGD}, F_D^{FGE}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ for $F_D^{GFE}, F_E^{GFE}, F_D^{GCE}, F_E^{GHD}, F_D^{DGF}, F_D^{DFG}, F_E^{DFG}, F_D^{EGF}, F_E^{FGD}, F_E^{FGE}, F_E^{CGD}, F_D^{HGE}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for $F_D^{GCD}, F_E^{GCE}, F_D^{GHD}, F_E^{GHE}, F_D^{DGC}, F_D^{DGH}, F_D^{DCG}, F_D^{DHG}, F_E^{EGC}, F_E^{EGH}, F_E^{ECG}, F_E^{EHG}, F_D^{CGD}, F_E^{CGE}, F_D^{HGD}, F_E^{HGE}$

- $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$ for $F_E^{GCD}, F_D^{GHE}, F_E^{DGH}, F_E^{DCG}, F_D^{EGC}, F_D^{EHG}, F_D^{CGE}, F_E^{HGD}$

$C_1C_3C_3$ - type:

- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for $F_G^{GFF}, F_G^{GCC}, F_G^{GHH}, F_G^{FFG}, F_G^{CCG}, F_G^{HHG}$
- -1 for $F_B^{GCH}, F_B^{GHF}, F_B^{GHC}, F_B^{FGC}, F_B^{FHG}, F_B^{CGF}, F_B^{CGH}, F_B^{CHG}, F_B^{HGC}, F_B^{HCG}$
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for $F_G^{FGF}, F_G^{CGC}, F_G^{HGH}$

$C_2C_2C_2$ - type:

- $\frac{1}{3} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & -1 & -1 & 2 \\ \sqrt{2} & -1 & 2 & -1 & -1 \\ \sqrt{2} & -1 & -1 & 2 & -1 \\ \sqrt{2} & 2 & -1 & -1 & -1 \end{pmatrix}$ for F_D^{DDD}, F_E^{EEE}
- $\frac{1}{\sqrt{3}} \begin{pmatrix} -1 & -1 & 1 & 0 \\ -1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix}$ for $F_E^{DDD}, F_D^{DDE}, F_D^{DED}, F_D^{EDD}$
- $\frac{1}{3} \begin{pmatrix} 1 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & -1 & 1 & -2 \\ \sqrt{2} & 1 & 2 & 1 & 1 \\ \sqrt{2} & 1 & -1 & -2 & 1 \\ \sqrt{2} & -2 & -1 & 1 & 1 \end{pmatrix}$ for F_E^{DDE}, F_D^{EED}
- $\frac{1}{3} \begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 & 1 & 1 & -2 \\ \sqrt{2} & 1 & -2 & 1 & 1 \\ \sqrt{2} & 1 & 1 & -2 & 1 \\ \sqrt{2} & -2 & 1 & 1 & 1 \end{pmatrix}$ for F_E^{DED}, F_D^{EDE}
- $\frac{1}{3} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 1 & 1 & 1 & -2 \\ \sqrt{2} & -1 & 2 & -1 & -1 \\ -\sqrt{2} & 1 & 1 & -2 & 1 \\ -\sqrt{2} & -2 & 1 & 1 & 1 \end{pmatrix}$ for F_D^{DEE}, F_E^{EDD}
- $\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 \end{pmatrix}$ for $F_E^{DEE}, F_E^{EDE}, F_E^{EED}, F_D^{EEE}$

$C_2C_2C_3$ - type:

- -1 for $F_B^{DDF}, F_B^{DCE}, F_B^{DHE}, F_B^{EDF}, F_B^{ECD}, F_B^{EHD}, F_B^{FDD}, F_B^{FDE}$

- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ for $F_G^{DDDF}, F_H^{DDDF}, F_G^{DDDC}, F_H^{DDDC}, F_F^{DDDH}, F_C^{DDDH}, F_F^{DEH}, F_G^{EDF}, F_H^{EDF}, F_H^{EEC}, F_C^{EEH}, F_G^{FED}, F_H^{FED}, F_G^{FEE}, F_H^{FEE}, F_F^{FEE}, F_C^{CDE}, F_G^{CED}, F_C^{CED}, F_G^{HDE}, F_H^{HDE}, F_H^{HDE}, F_H^{HEE}, F_F^{HEE}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for $F_F^{DDF}, F_C^{DDF}, F_G^{DDC}, F_F^{DDC}, F_C^{DDC}, F_G^{DDH}, F_H^{DDH}, F_F^{DEC}, F_C^{EDF}, F_G^{EEC}, F_C^{EEC}, F_G^{EEH}, F_H^{EEH}, F_F^{FDD}, F_C^{FDD}, F_C^{FDE}, F_G^{CDD}, F_C^{CDD}, F_C^{CDD}, F_F^{CED}, F_G^{CEE}, F_C^{CEE}, F_G^{HDD}, F_H^{HDD}, F_G^{HEE}, F_H^{HEE}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ for $F_G^{DEF}, F_H^{DEF}, F_G^{DEC}, F_C^{DEC}, F_H^{DEH}, F_C^{EDC}, F_G^{EDH}, F_F^{EDH}, F_H^{EDH}, F_G^{EEF}, F_F^{EEF}, F_H^{EEF}, F_F^{FDD}, F_H^{FDD}, F_G^{FDE}, F_H^{FDE}, F_C^{CDD}, F_H^{CEE}, F_H^{HDD}, F_C^{HDD}, F_F^{HED}, F_C^{HEE}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ for $F_F^{DEF}, F_H^{DEC}, F_F^{EDF}, F_C^{EDH}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ for $F_C^{DEF}, F_C^{DEH}, F_F^{EDC}, F_H^{EDC}, F_C^{EEF}, F_F^{EEC}, F_C^{FED}, F_C^{FEE}, F_F^{CDE}, F_H^{CDE}, F_C^{CEE}, F_C^{HED}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$ for $F_G^{DEH}, F_G^{EDC}, F_G^{CDE}, F_G^{HED}$
- $\frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$ for $F_G^{DFD}, F_C^{DFD}, F_H^{DFD}, F_F^{DCD}, F_H^{DCD}, F_F^{DHD}, F_C^{DHD}$
- $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for $F_F^{DFD}, F_C^{DCD}, F_G^{DHD}$
- $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{pmatrix}$ for $F_G^{DFE}, F_C^{DFE}, F_H^{DFE}, F_F^{ECD}, F_F^{EHD}$
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for F_F^{DFE}, F_F^{EFD}
- $\frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$ for F_G^{DCD}, F_H^{DHD}
- $\frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}$ for $F_G^{DCE}, F_H^{DHE}, F_G^{ECD}, F_H^{EHD}$
- $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & -1 \\ 1 & -\sqrt{3} \end{pmatrix}$ for $F_F^{DCE}, F_F^{DHE}, F_G^{EFD}, F_C^{EFD}, F_H^{EFD}$
- $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ for $F_C^{DCE}, F_G^{DHE}, F_C^{ECD}, F_G^{EHD}$

- $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$ for $F_H^{DCE}, F_C^{DHE}, F_H^{ECD}, F_C^{EHD}$
- $\frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$ for $F_G^{EFE}, F_C^{EFE}, F_H^{EFE}, F_G^{ECE}, F_F^{ECE}, F_F^{EHE}, F_H^{EHE}$
- $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ for $F_F^{EFE}, F_C^{ECE}, F_G^{EHE}$
- $\frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$ for F_H^{ECE}, F_C^{EHE}
- $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$ for $F_F^{FDE}, F_F^{FED}, F_H^{CED}, F_C^{HDE}$

$C_2C_3C_3$ - type:

- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for $F_D^{DFD}, F_D^{DFC}, F_E^{DFC}, F_D^{DCF}, F_D^{DCC}, F_D^{DHH}, F_D^{ECF}, F_E^{ECC}, F_E^{EHH}, F_D^{FFD}, F_D^{FCD}, F_D^{FCE}, F_D^{CFD}, F_E^{CFD}, F_D^{CCD}, F_E^{CCE}, F_D^{HHD}, F_E^{HHE}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$ for $F_E^{DFD}, F_E^{DHC}, F_D^{EFF}, F_D^{ECH}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ for $F_D^{DFH}, F_E^{DFH}, F_D^{DCH}, F_D^{DHF}, F_D^{DHC}, F_E^{ECH}, F_D^{EHF}, F_E^{EHC}, F_E^{FFE}, F_E^{FHD}, F_E^{FHE}, F_E^{CCD}, F_D^{CCE}, F_D^{HFE}, F_E^{HFE}, F_E^{HHD}, F_D^{HHE}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ for $F_E^{DCF}, F_E^{DCH}, F_D^{EFC}, F_E^{EFC}, F_E^{ECF}, F_D^{EHC}, F_E^{FCD}, F_E^{FCE}, F_D^{CFE}, F_E^{CFE}, F_D^{CHE}, F_E^{HCD}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ for $F_E^{DCC}, F_E^{DHF}, F_D^{DHH}, F_E^{EFF}, F_D^{EFH}, F_E^{EFH}, F_D^{ECC}, F_E^{EHF}, F_D^{EHH}, F_D^{FHD}, F_D^{FHE}, F_D^{CHD}, F_E^{CHE}, F_D^{HFD}, F_E^{HFD}, F_D^{HCD}, F_E^{HCE}$
- $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for F_D^{FDF}, F_D^{CDC}
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for F_E^{FDF}, F_D^{FEF}
- $\frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$ for $F_D^{FDC}, F_D^{FDH}, F_D^{CDF}, F_D^{CDH}, F_D^{HDF}, F_D^{HDC}$
- $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{pmatrix}$ for $F_E^{FDC}, F_E^{FDH}, F_D^{CEF}, F_D^{HEF}$
- $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ for F_E^{FEF}, F_E^{CEC}

- $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & -1 \\ 1 & -\sqrt{3} \end{pmatrix}$ for $F_D^{FEC}, F_D^{FEH}, F_E^{CDF}, F_E^{HDF}$
- $\frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$ for $F_E^{FEC}, F_E^{FEH}, F_E^{CEF}, F_E^{HEF}, F_E^{HEH}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ for $F_E^{FFD}, F_D^{FFE}, F_E^{CHD}, F_D^{HCE}$
- $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ for F_E^{CDC}, F_D^{CEC}
- $\frac{1}{2} \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$ for $F_E^{CDH}, F_D^{CEH}, F_E^{HDC}, F_D^{HEC}$
- $\frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$ for F_E^{CEH}, F_E^{HEC}
- $\frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$ for F_D^{HDD}
- $\frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}$ for F_E^{HDD}, F_D^{HEH}

$C_3C_3C_3$ - type:

- -1 for $F_B^{FFF}, F_B^{FCH}, F_B^{CCC}, F_B^{HCF}, F_B^{HHH}$
- $\frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$ for $F_F^{FFF}, F_C^{CCC}, F_H^{HHH}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ for $F_C^{FFC}, F_F^{CCF}, F_H^{CCH}, F_C^{HHC}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for $F_H^{FFH}, F_F^{FHH}, F_H^{HFF}, F_F^{HHF}$
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for $F_C^{FCF}, F_H^{FHF}, F_F^{CFC}, F_H^{CHC}, F_F^{HFF}, F_C^{FCH}$
- $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ for $F_F^{FCC}, F_C^{CFF}, F_C^{CHH}, F_H^{HCC}$

A. 2 The rest of R -symbols

Beside the R -symbols at the beginning of the section, the rest are:

- 1 for $R_A^{BB}, R_F^{GH}, R_C^{GH}, R_D^{DF}, R_D^{DC}, R_E^{EF}, R_E^{EC}, R_D^{FD}, R_E^{FE}, R_A^{FF}, R_F^{FF}, R_D^{CD}, R_E^{CE}, R_A^{CC}, R_C^{CC}, R_F^{HG}, R_C^{HG}, R_A^{EE}, R_F^{EE}, R_C^{EE}$
- -1 for $R_B^{GG}, R_F^{BF}, R_C^{BC}, R_H^{BH}, R_G^{GB}, R_F^{FB}, R_B^{FF}, R_C^{CB}, R_B^{CC}, R_H^{HB}, R_A^{DD}, R_F^{DD}, R_C^{DD}$
- i for $R_E^{BD}, R_E^{DB}, R_D^{EF}, R_D^{EC}, R_D^{FE}, R_D^{CE}, R_B^{DE}, R_B^{ED}$
- $-i$ for $R_D^{BE}, R_E^{DF}, R_E^{DC}, R_D^{EB}, R_E^{FD}, R_E^{CD}, R_F^{DE}, R_C^{DE}, R_F^{ED}, R_C^{ED}$
- ω^2 for $R_A^{GG}, R_H^{GF}, R_H^{GC}, R_D^{DH}, R_E^{EH}, R_H^{FG}, R_G^{FC}, R_C^{FH}, R_C^{CG}, R_G^{CF}, R_F^{CH}, R_D^{HD}, R_E^{HE}, R_C^{HF}, R_F^{HC}, R_H^{HH}, R_G^{EE}$

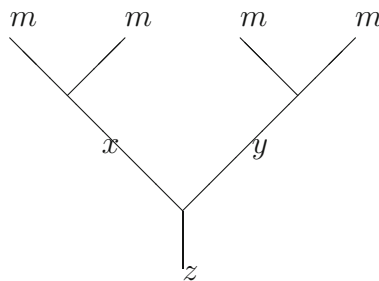
- $-\omega^2$ for R_B^{GG}, R_G^{DD}
- ω for $R_G^{GG}, R_D^{GD}, R_E^{GE}, R_C^{GF}, R_F^{GC}, R_D^{DG}, R_E^{EG}, R_C^{FG}, R_H^{FC}, R_G^{FH}, R_F^{CG}, R_H^{CF}, R_G^{CH}, R_G^{HF}, R_G^{HC}, R_A^{HH}, R_H^{EE}$
- $-\omega$ for R_H^{DD}, R_B^{HH}
- ωi for R_D^{GE}, R_D^{EG}
- $-\omega i$ for $R_E^{GD}, R_E^{DG}, R_H^{DE}, R_H^{ED}$
- $\omega^2 i$ for R_D^{EH}, R_D^{HE}
- $-\omega^2 i$ for $R_G^{DE}, R_G^{ED}, R_E^{DH}, R_E^{HD}$

Appendix 2: Representations of braid group \mathcal{B}_4

In this appendix, mathematically, we study whether or not the representations of \mathcal{B}_4 are irreducible and identify the images of those representation on each irreducible summand. We will refer to each irreducible summand as a *sector*. For our application to anyonic quantum computation, we also determine whether or not there are unitary transformations (braiding quantum circuits) in the images that are powerful for quantum computation, especially whether or not these circuits lead to a universal gate set.

We will provide explicitly the braiding matrices for $\sigma_1, \sigma_2, \sigma_3$ and then compute what is the group generated by them. Without loss of generality, we may multiply the σ_i 's by a common factor so that they all have determinant = 1 (Note that all the σ_i 's are conjugate to each other). We still denote the new representation by $\rho(m, z)$. We will focus on sectors which are three-dimensional. In this case, the images of the representation on such sectors are subgroups of $SU(3)$. As will be seen later, some interesting subgroups of $SU(3)$ will arise as the image.

As in Sect. 2, the representations are denoted by $\rho(m, z)$ on the space V_z^{mmmm} , which corresponds to the following splitting tree:



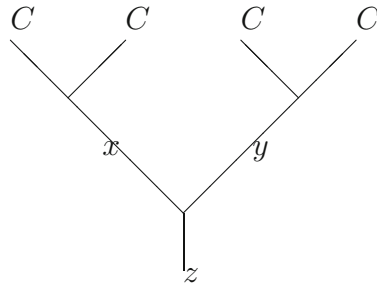
Our results are summarized in Tables 3 and 4 in Sect. 2. Now we examine each representation explicitly in the following subsections.

Remark 2 The matrices σ_i 's depend on the fusion rules of the two anyons m , the $6j$ -symbols and the R -symbols R_x^{mm} . From this point of view, the anyons A and B are not interesting because their R -matrices are trivial. It follows that their representations are also projectively trivial. Also, the anyons C and F have identical R -matrices. D

and E are identical if we multiply the R -matrices of D by -1 . Similarly, G and H are identical if we replace $\omega = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$ in the R -matrices of G by its complex conjugate. Therefore, it suffices to consider the cases where the anyon m is C, D , and G .

B. 1 Representations on V_z^{CCCC}

There are three choices for z that make the following splitting tree admissible, namely A, B or C .



B. 1.1 $z = A$

The basis of V_A^{CCCC} is $\{|AA\rangle, |BB\rangle, |CC\rangle\}$. Under this basis,

$$\sigma_1 = \sigma_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \sigma_2 = -\frac{1}{2} \begin{pmatrix} 1 & -1 & \sqrt{2} \\ -1 & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$$

This representation splits into two sectors S_1 and S_2 , where S_1 is a 1-dim irrep mapping σ_i to -1 and S_2 is a 2-dim irrep spanned by $\{|BB\rangle, \frac{\sqrt{3}}{3}|AA\rangle - \frac{\sqrt{6}}{3}|CC\rangle\}$. The matrices of the σ_i 's under the basis of S_2 are given by:

$$\sigma_1 = \sigma_3 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_2 = \frac{i}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

They generate a group which is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_4$.

B. 1.2 $z = B$

The Hilbert space V_B^{CCCC} is also three-dimensional with basis $\{|CC\rangle, |AB\rangle, |BA\rangle\}$. The matrices of the σ_i 's are given by:

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 = -\frac{1}{2} \begin{pmatrix} 0 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 1 & -1 \\ -\sqrt{2} & -1 & 1 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The representation is irreducible with image isomorphic to the permutation group \mathcal{S}_4 .

B. 1.3 $z = C$

The Hilbert space V_C^{CCCC} is five-dimensional with basis $\{|CC\rangle, |AC\rangle, |CA\rangle, |BC\rangle, |CB\rangle\}$. And the image of the σ_i 's is given by :

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

V_C^{CCCC} splits into the direct sum of two trivial irreps and a 3-dim irrep V . V has a basis $\{\frac{1}{\sqrt{2}}(|AC\rangle - |CA\rangle), |BC\rangle, |CB\rangle\}$. Under this basis, the σ_i 's have the following image:

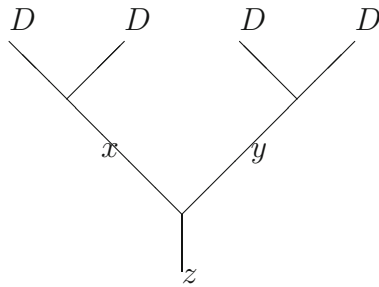
$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \sigma_2 = -\frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \\ -\sqrt{2} & 1 & 1 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

And they generate a group which is also isomorphic to \mathcal{S}_4 .

Therefore, if we braid four anyons C , then all the images of the representations are very small.

B. 2 Representations on V_z^{DDDD}

There are six choices for z , namely A, B, C, F, G and H . By Remark 2, we only need to consider cases where $z = A, B, F$ and G .



B. 2. $1 z = A$

The space V_A^{DDDD} is five-dimensional with basis $\{|AA\rangle, |GG\rangle, |FF\rangle, |CC\rangle, |HH\rangle\}$. Under this basis, the matrices of the σ_i 's are as follows:

$$\sigma_1 = \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \omega \end{pmatrix}$$

$$\sigma_2 = \frac{1}{6} \begin{pmatrix} 2 & \sqrt{2} + \sqrt{6}i & 2\sqrt{2} & 2\sqrt{2} & -\sqrt{2} - \sqrt{6}i \\ -\sqrt{2} + \sqrt{6}i & 1 + \sqrt{3}i & 1 - \sqrt{3}i & 1 - \sqrt{3}i & 4 \\ 2\sqrt{2} & 1 - \sqrt{3}i & 4 & -2 & 1 + \sqrt{3}i \\ 2\sqrt{2} & 1 - \sqrt{3}i & -2 & 4 & 1 + \sqrt{3}i \\ -\sqrt{2} - \sqrt{6}i & 4 & 1 + \sqrt{3}i & 1 + \sqrt{3}i & 1 - \sqrt{3}i \end{pmatrix}$$

where $\omega = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$ is a third root of unity.

This representation splits into the direct sum of two trivial representations and a three-dimensional sector S , which is spanned by the basis $\{|GG\rangle, |HH\rangle, \frac{1}{2}(-\sqrt{2}|AA\rangle + |CC\rangle + |FF\rangle)\}$. The representation on S is generated by the following matrices:

$$\sigma_1 = \sigma_3 = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \sigma_2 = \frac{1}{6} \begin{pmatrix} 1 + \sqrt{3}i & 4 & 2 - 2\sqrt{3}i \\ 4 & 1 - \sqrt{3}i & 2 + 2\sqrt{3}i \\ 2 - 2\sqrt{3}i & 2 + 2\sqrt{3}i & -2 \end{pmatrix}$$

They generate a group of order 12 which is isomorphic to \mathcal{A}_4 , the alternating group.

B. 2. $2 z = B$

V_B^{DDDD} is four-dimensional with basis $\{|GG\rangle, |FF\rangle, |CC\rangle, |HH\rangle\}$. The matrices of the generators are as follows:

$$\sigma_1 = \sigma_3 = \begin{pmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

$$\sigma_2 = \frac{1}{6} \begin{pmatrix} 3 - \sqrt{3}i & 3 + \sqrt{3}i & 3 + \sqrt{3}i & 0 \\ 3 + \sqrt{3}i & 0 & -2\sqrt{3} & -3 + \sqrt{3}i \\ 3 + \sqrt{3}i & -2\sqrt{3} & 0 & 3 - \sqrt{3}i \\ 0 & -3 + \sqrt{3}i & 3 - \sqrt{3}i & 3 + \sqrt{3}i \end{pmatrix}$$

This representation splits into the sum of two two-dimensional sectors S_1 and S_2 , where S_1 is spanned by $\{|GG\rangle, \frac{1}{\sqrt{2}}(|CC\rangle + |FF\rangle)\}$, and S_2 is spanned by $\{|HH\rangle, \frac{1}{\sqrt{2}}(|FF\rangle - |CC\rangle)\}$.

On the sector S_1 , the generators have matrices:

$$\sigma_1 = \sigma_3 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{3}i}{6} & -\frac{\sqrt{6}i}{3} \\ -\frac{\sqrt{6}i}{3} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} \end{pmatrix}$$

They generate a group of size 24 which is isomorphic to $SL(2, \mathbb{F}_3)$. Modulo the center, we get \mathcal{A}_4 .

The representation on the other sector S_2 is exactly the same of that on S_1 .

B. 2. $3z = F$

The space V_F^{DDDD} is nine-dimensional and has a basis $\{|FF\rangle, |AF\rangle, |FA\rangle, |GC\rangle, |CG\rangle, |GH\rangle, |HG\rangle, |CH\rangle, |HC\rangle\}$. The generators of \mathcal{B}_4 have the following matrices:

$$\sigma_1 = \begin{pmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & \omega^2 & & & & & \\ & & & & 1 & & & & \\ & & & & & \omega^2 & & & \\ & & & & & & \omega & & \\ & & & & & & & 1 & \\ & & & & & & & & \omega \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \omega^2 & & & & \\ & & & & & \omega & & & \\ & & & & & & \omega^2 & & \\ & & & & & & & \omega & \\ & & & & & & & & 1 \end{pmatrix}$$

$$\sigma_2 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & \omega^2 & \omega^2 & 1 & 1 & \omega & \omega \\ 1 & 1 & 1 & \omega & 1 & \omega & \omega^2 & 1 & \omega^2 \\ 1 & 1 & 1 & 1 & \omega & \omega^2 & \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 & 1 & \omega^2 & \omega & 1 & 1 & 1 \\ \omega^2 & 1 & \omega & \omega^2 & 1 & 1 & \omega & 1 & 1 \\ 1 & \omega & \omega^2 & \omega & 1 & 1 & 1 & \omega^2 & 1 \\ 1 & \omega^2 & \omega & 1 & \omega & 1 & 1 & 1 & \omega^2 \\ \omega & 1 & \omega^2 & 1 & 1 & \omega^2 & 1 & 1 & \omega \\ \omega & \omega^2 & 1 & 1 & 1 & 1 & \omega^2 & \omega & 1 \end{pmatrix}$$

This representation splits into the sum of a 1-dim trivial representation and an 8-dim irrep. The 1-dim irrep is spanned by the element $\frac{1}{\sqrt{3}}(|FF\rangle + |AF\rangle + |FA\rangle)$. The 8-dim irrep has an image in $U(8)$ of size 216 which is isomorphic to the famous Hessian group $\Sigma(216)$ in physics literature. The following is a presentation of $\Sigma(216)$ from analyzing the matrices of the generators σ_i :

$$\langle a, b, c | aba = bab, bcb = cbc, ac = ca, a^3 = (ab)^6 = (bc)^6 = (abcaba)^2 = 1 \rangle.$$

As an abstract group, it is isomorphic to $((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8) \times \mathbb{Z}_3$, where Q_8 is the quaternion group of order 8.

We will see this group again later.

B. 2. $4z = G$

The space V_G^{DDDD} is also nine-dimensional with a basis $\{|GG\rangle, |AG\rangle, |GA\rangle, |FC\rangle, |CF\rangle, |FH\rangle, |HF\rangle, |CH\rangle, |HC\rangle\}$. As always, we first look at the matrices of the σ_i 's:

$$\sigma_1 = \begin{pmatrix} \omega^2 & & & & & & & & \\ & 1 & & & & & & & \\ & & \omega^2 & & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & \omega & & \\ & & & & & & & 1 & \\ & & & & & & & & \omega \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} \omega^2 & & & & & & & & \\ & \omega^2 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & & & & \\ & & & & & \omega & & & \\ & & & & & & 1 & & \\ & & & & & & & \omega & \\ & & & & & & & & 1 \end{pmatrix}$$

$$\sigma_2 = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega & \omega^2 & \omega^2 & 1 & 1 & 1 & 1 \\ \omega & 1 & \omega & 1 & 1 & 1 & \omega^2 & 1 & \omega^2 \\ \omega & \omega & 1 & 1 & 1 & \omega^2 & 1 & \omega & \omega \\ \omega^2 & 1 & 1 & 1 & \omega^2 & 1 & \omega & 1 & \omega \\ \omega^2 & 1 & 1 & \omega^2 & 1 & \omega & 1 & 1 & \omega \\ 1 & 1 & \omega^2 & 1 & \omega & 1 & 1 & \omega^2 & \omega \\ 1 & \omega^2 & 1 & \omega & 1 & 1 & 1 & \omega & \omega^2 \\ 1 & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & 1 \\ 1 & \omega^2 & 1 & 1 & \omega & \omega & \omega^2 & 1 & 1 \end{pmatrix}$$

The representation splits into the sum of a 6-dim irrep and a 3-dim irrep.

Denote this 3-dim irrep by W , which is spanned by the basis $\{\frac{1}{\sqrt{2}}(|FC\rangle - |CF\rangle), \frac{1}{\sqrt{2}}(|CH\rangle - |FH\rangle), \frac{1}{\sqrt{2}}(|HF\rangle - |HC\rangle)\}$. If we use this subspace W as computational space, we will have a qutrit. The three basis elements above correspond to $|0\rangle, |1\rangle, |2\rangle$.

Under this basis, the matrices of the σ_i 's (unnormalized) on W are as follows:

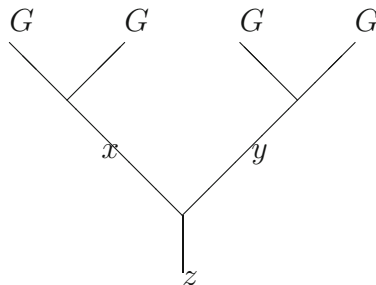
$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}i}{6} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} \\ -\frac{1}{2} + \frac{\sqrt{3}i}{6} & \frac{1}{2} + \frac{\sqrt{3}i}{6} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} \\ -\frac{1}{2} + \frac{\sqrt{3}i}{6} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} & \frac{1}{2} + \frac{\sqrt{3}i}{6} \end{pmatrix}$$

The group generated by them has order 648 with a center of size 3. The elements in the center are scalar matrices. And the group modulo center has order 216 which is isomorphic to the Hessian group $\Sigma(216)$.

B. 3 Representations on V_z^{GGGG}

The possible choices of z are A, B and G .



B. 3. $Iz = A$

The space V_A^{GGGG} is three-dimensional with a basis $\{|AA\rangle, |BB\rangle, |GG\rangle\}$. Under this basis, the matrices for the generators σ_i 's are given by

$$\sigma_1 = \sigma_3 = \tau \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

$$\sigma_2 = \tau \begin{pmatrix} \frac{1}{2}\omega & -\frac{1}{2}\omega & \frac{1}{\sqrt{2}}\omega^2 \\ -\frac{1}{2}\omega & \frac{1}{2}\omega & \frac{1}{\sqrt{2}}\omega^2 \\ \frac{1}{\sqrt{2}}\omega^2 & \frac{1}{\sqrt{2}}\omega^2 & 0 \end{pmatrix}$$

where $\tau = e^{-\frac{\pi i}{9}}$.

This representation is irreducible, and the group generated by them has a structure of $(\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathcal{S}_3$ with order 162, which is isomorphic to the group $D(9, 1, 1; 2, 1, 1)$.

We recall the definition of $D(n, a, b; d, r, s)$ below. For more information about this type of subgroups of $SU(3)$, see [12].

Let

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad F = F(n, a, b) = \begin{pmatrix} e^{\frac{2\pi ia}{n}} & 0 & 0 \\ 0 & e^{\frac{2\pi ib}{n}} & 0 \\ 0 & 0 & e^{\frac{2\pi i(-a-b)}{n}} \end{pmatrix}$$

$$G = G(d, r, s) = \begin{pmatrix} e^{\frac{2\pi ir}{d}} & 0 & 0 \\ 0 & 0 & e^{\frac{2\pi is}{d}} \\ 0 & -e^{\frac{2\pi i(-r-s)}{d}} & 0 \end{pmatrix}$$

Then $D(n, a, b; d, r, s) := \langle E, F(n, a, b), G(d, r, s) \rangle$.

Actually one can show that the group generated by the σ_i 's is isomorphic to $D(9, 1, 1; 2, 1, 1)$ via a conjugation by some unitary matrix.

B. 3. $2z = B$

V_B^{GGGG} is also three-dimensional with a basis $\{|GG\rangle, |AB\rangle, |BA\rangle\}$. The matrices of the σ_i 's are given by

$$\sigma_1 = \tau \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & -\omega^2 \end{pmatrix} \quad \sigma_3 = \tau \begin{pmatrix} \omega & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$$\sigma_2 = \tau \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}}\omega^2 & -\frac{1}{\sqrt{2}}\omega^2 \\ -\frac{1}{\sqrt{2}}\omega^2 & \frac{1}{2}\omega & -\frac{1}{2}\omega \\ -\frac{1}{\sqrt{2}}\omega^2 & -\frac{1}{2}\omega & \frac{1}{2}\omega \end{pmatrix}$$

where $\tau = e^{-\frac{\pi i}{9}}$.

Again this representation is irreducible, and they generate a group with structure $(\mathbb{Z}_{18} \times \mathbb{Z}_6) \rtimes S_3$, which is isomorphic to the group $D(18, 1, 1; 2, 1, 1)$ [12]. So it has order 648.

Let

$$p = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Direct calculations show that conjugation by the matrix p gives the isomorphism from our group generated by the σ_i 's to $D(18, 1, 1; 2, 1, 1)$.

B. 3. $3z = G$

V_G^{GGG} is now five-dimensional with a basis $\{|GG\rangle, |AG\rangle, |GA\rangle, |BG\rangle, |GB\rangle\}$. The matrices of the generators σ_i 's are as follows:

$$\sigma_1 = \begin{pmatrix} \omega & 0 & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & -\omega^2 & 0 \\ 0 & 0 & 0 & 0 & \omega \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} \omega & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\omega & \frac{1}{2}\omega^2 & \frac{1}{2}\omega & -\frac{1}{2}\omega^2 \\ 0 & \frac{1}{2}\omega^2 & \frac{1}{2}\omega & -\frac{1}{2}\omega^2 & \frac{1}{2}\omega \\ 0 & \frac{1}{2}\omega & -\frac{1}{2}\omega^2 & \frac{1}{2}\omega & \frac{1}{2}\omega^2 \\ 0 & -\frac{1}{2}\omega^2 & \frac{1}{2}\omega & \frac{1}{2}\omega^2 & \frac{1}{2}\omega \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} \omega & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & -\omega^2 \end{pmatrix}$$

It is obvious that $|GG\rangle$ is a common eigenvector of the σ_i 's. So it spans a 1-dim irrep of \mathcal{B}_4 . The orthogonal complement spanned by the other 4 basis elements is a 4-dim irrep.

The group generated by the σ_i 's has order 648. And GAP shows that it has a structure of $((\mathbb{Z}_3 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2)) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3 \rtimes \mathbb{Z}_2$.

References

1. Anwar, H., Campbell, E.T., Browne, D.E.: Qutrit magic state distillation. *New J. Phys.* **14**(6), 063006 (2012)
2. Barkeshli, M., Jian, C.-M., Qi, X.-L.: Twist defects and projective non-abelian braiding statistics. *Phys. Rev. B* **87**(4), 045130 (2013)
3. Beigi, S., Shor, P.W., Whalen, D.: The quantum double model with boundary: condensations and symmetries. *Commun. Math. Phys.* **306**(3), 663–694 (2011)
4. Bruillard, P., Ng, S.-H., Rowell, E., Wang, Z.: On modular categories (2013). [arXiv:1310.7050v1](https://arxiv.org/abs/1310.7050v1)
5. Etingof, P., Rowell, E., Witherspoon, S.: Braid group representations from twisted quantum doubles of finite groups. *Pacific journal of mathematics* **234**(1), 33–42 (2008)
6. Hastings, M.B., Nayak, C., Wang, Z.: Metaplectic anyons, majorana zero modes, and their computational power. *Physical Review B* **87**(16), 165421 (2013)
7. Hastings, M.B., Nayak, C., Wang, Z.: On metaplectic modular categories and their applications (2013). [arXiv:1303.1202](https://arxiv.org/abs/1303.1202)
8. Howard, M., Vala, J.: Qudit versions of the qubit $\pi/8$ gate. *Phys. Rev. A* **86**(2), 022316 (2012)
9. Kitaev, A.Y.: Private communication. See also quantum computation, ph/cs 219, winter 2007, problem set 3
10. Kitaev, A.Y.: Unpaired majorana fermions in quantum wires. *Phys. Usp.* **44**(10S), 131 (2001)
11. Kitaev, A.Y.: Fault-tolerant quantum computation by anyons. *Ann. Phys.* **303**(1), 2–30 (2003)
12. Ludl, P.O.: Comments on the classification of the finite subgroups of SU (3). *J. Phys. A Math. Theor.* **44**(25), 255204 (2011)
13. Mochon, C.: Anyons from nonsolvable finite groups are sufficient for universal quantum computation. *Phys. Rev. A* **67**(2), 022315 (2003)
14. Mochon, C.: Anyon computers with smaller groups. *Phys. Rev. A* **69**(3), 032306 (2004)
15. Mong, R.S.K., Clarke, D.J., Alicea, J., Lindner, N.H., Fendley, P., Nayak, C., Oreg, Y., Stern, A., Berg, E., Shtengel, K., et al.: Universal topological quantum computation from a superconductor/abelian quantum hall heterostructure (2013). [arXiv:1307.4403](https://arxiv.org/abs/1307.4403)
16. Naidu, D., Rowell, E.C.: A finiteness property for braided fusion categories. *Algebr. Represent. Theory* **14**(5), 837–855 (2011)
17. Nayak, C., Simon, S.H., Stern, A., Freedman, M., Sarma, S.D.: Non-abelian anyons and topological quantum computation. *Rev. Mod. Phys.* **80**(3), 1083 (2008)
18. Ogburn, W.R., Preskill, J.: Topological quantum computation. In: *Quantum Computing and Quantum Communications*, pp. 341–356. Springer, Berlin (1999)
19. Read, N., Rezayi, E.: Beyond paired quantum hall states: parafermions and incompressible states in the first excited landau level. *Phys. Rev. B* **59**(12), 8084 (1999)
20. Siehler, J.: Near-group categories. *Algeb. Geom. Topol.* **3**, 719–775 (2003)
21. Wang, Z.: *Topological Quantum Computation*, vol. 112. American Mathematical Society, Providence (2010)