

On generalized symmetries and structure of modular categories

Shawn Xingshan Cui^{1,2}, Modjtaba Shokrian Zini³ & Zhenghan Wang^{3,4,*}

¹Stanford Institute for Theoretical Physics, Stanford University, Stanford, CA 94305, USA;

²Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA;

³Department of Mathematics, University of California at Santa Barbara, Santa Barbara, CA 93106, USA;

⁴Microsoft Station Q, Santa Barbara, CA 93106, USA

Email: xingshan@vt.edu, shokrian@math.ucsb.edu, zhenghwa@microsoft.com

Received September 11, 2018; accepted December 13, 2018; published online January 11, 2019

Abstract Pursuing a generalization of group symmetries of modular categories to category symmetries in topological phases of matter, we study linear Hopf monads. The main goal is a generalization of extension and gauging group symmetries to category symmetries of modular categories, which include also categorical Hopf algebras as special cases. As an application, we propose an analogue of the classification of finite simple groups to modular categories, where we define simple modular categories as the prime ones without any nontrivial normal algebras.

Keywords modular category, category symmetry, Hopf monad, tensor functor

MSC(2010) 18D10, 81R05

Citation: Cui S X, Zini M S, Wang Z. On generalized symmetries and structure of modular categories. *Sci China Math*, 2019, 62: 417–446, <https://doi.org/10.1007/s11425-018-9455-5>

1 Introduction

Group symmetries are central in mathematics and physics. But the group symmetry alone is not adequate to capture phenomena such as symmetry fractionalization, symmetry defects, and gauging in topological phases of matter, where categorical group symmetries are used [2, 12]. In this paper, we propose a further generalization from categorical groups to fusion categories using linear Hopf monads, and investigate the basic properties for such generalized symmetries. The main goal is a generalization of extension and gauging group symmetries to category symmetries of modular categories, which include also categorical Hopf algebras as special cases. As an application, we propose an analogue of the classification of finite simple groups to modular categories, where we define simple modular categories as the prime ones without any nontrivial normal algebras.

Exemplary examples of group symmetries in the mathematical formulation of quantum field theories include $U(1)$, $SU(2)$ and $SU(3)$ in the standard model. Quantum groups, loosely defined as Hopf algebras, appeared in the study of inverse scattering problems in mathematical physics and are instrumental in understanding topological phases of matter, which lie outside the Landau paradigm of phases of matter

* Corresponding author

based on group symmetries. Topological phases of matter in two spatial dimensions are modeled by anyon models, mathematically unitary modular categories. Finite group symmetries of anyon models have been extensively studied recently revealing a deep interplay between symmetry and topological order through symmetry fractionalization, defects, and gauging [2, 12]. How to extend symmetries beyond finite groups arises as a natural problem. Our generalization to linear Hopf monads as symmetries of anyon models and unitary fusion categories can be equivalently formulated as fusion category symmetries of modular categories.

A linear Hopf monad is a far-reaching generalization of a Hopf algebra [7, 9], which is closely related to adjunctions of tensor functors. A Hopf monad $T : \mathcal{C} \rightarrow \mathcal{C}$ of a fusion category \mathcal{C} is essentially the same as a tensor functor $D : \mathcal{C}^T \rightarrow \mathcal{C}$, where \mathcal{C}^T , assuming T is semisimple, is the *fusion* category of Eilenberg-Moore category of T -modules. Two large classes of Hopf monads are those from group actions G on \mathcal{C} and categorical Hopf algebras \mathcal{H} in \mathcal{C} .

One of the motivations for this work is the possibility of generalizing the doubled Haagerup category by gauging some Hopf monad symmetries on abelian modular categories¹⁾. It is unclear at the moment if this approach is a viable one.

The contents of the paper are as follows. In Section 3, we first recall the definition of Hopf monad [7] that does not follow the standard algebra-coalgebra one [9] using the so-called fusion functors. In Section 4, we skeletalize the categorical definition of Hopf monads, which are useful for concrete calculation. In Section 5, we attempt to characterize Hopf monads on $\text{Vec}_{\mathbb{Z}_p}$. In Section 6, we outline the theory of extension, gauging, and generalized symmetry defects. Finally, we propose a structure theory for modular categories.

2 Background material and a list of notations

Fusion categories and braided fusion categories in this paper are assumed to be unitary over \mathbb{C} . Therefore, all fusion categories are spherical. All Hopf monads (HM) are \mathbb{C} -linear and semisimple ensuring their Eilenberg-Moore module categories to be fusion [9, Remark 6.2]. We follow the book [15] for the various notions of categories such as abelian, monoidal, braided, unitary, and fusion, and similarly for notions of functors with the additional constraint that all functors considered will be linear. Also, there will be an exception: A monoidal functor F has maps

$$F_2(X, Y) : F(X) \otimes F(Y) \rightarrow F(X \otimes Y), \quad F_0 : \mathbf{1} \rightarrow F(\mathbf{1}),$$

which are *not* necessarily isomorphisms. If they are, the functor will be called *strong* monoidal. As a result, a tensor functor is a (linear) strong monoidal functor which is exact and faithful. We shall also assume well-known facts about module categories (see, e.g., [15, Chapter 7] or [28]).

Notations. 1. $\mathcal{C}, \mathcal{D}, \mathcal{F}$: fusion categories.

2. \mathcal{A}, \mathcal{H} : algebras, usually condensable and Hopf, respectively.

3. \mathcal{B} : unitary braided fusion category.

4. $\mathcal{B}_{\mathcal{A}}$: the condensed fusion category from condensing an algebra \mathcal{A} of \mathcal{B} .

5. $\text{Endo}(\mathcal{C})$: the monoidal category of endo-functors of \mathcal{C} .

6. $\text{Modc}(\mathcal{C})$: the category of module categories of \mathcal{C} .

7. $\text{Bimodc}(\mathcal{C})$: the linear monoidal category of bimodule categories of \mathcal{C} .

8. $\text{Modc}(\mathcal{B})$: the linear monoidal category of module categories of \mathcal{B} regarded as bimodule categories using the braiding, not the same as $\text{Bimodc}(\mathcal{B})$.

9. $\text{Aut}_{\otimes}^{\text{br}}(\mathcal{B}), \underline{\text{Aut}}_{\otimes}^{\text{br}}(\mathcal{B}), \underline{\underline{\text{Aut}}}_{\otimes}^{\text{br}}(\mathcal{B})$: the group, 2-groupoid, 3-groupoid of braided equivalences of \mathcal{B} .

10. Similarly, $\text{Pic}(\mathcal{B}), \underline{\text{Pic}}(\mathcal{B}), \underline{\underline{\text{Pic}}}(\mathcal{B})$: the group, 2-groupoid, 3-groupoid of invertible module categories of \mathcal{B} .

¹⁾ The third author would like to thank Colleen Delaney, Terry Gannon and James Tener for collaborating on this project.

11. \mathcal{C}^T : the fusion category of T -modules of a Hopf monad $T \in \text{Endo}(\mathcal{C})$, with objects $(M, r), M \in \mathcal{C}$, and $r : T(M) \rightarrow M$ the T -action.
12. \mathcal{B}_T^\times : T -crossed braided category extended over \mathcal{B} .
13. $\mathcal{B}_T^{\times, T}$: the gauging of \mathcal{B} .
14. $\underline{\rho}$: categorical group homomorphism.
15. $\underline{\underline{\rho}}$: categorical 2-group homomorphism.
16. $\mathcal{Z}(\mathcal{C})$ and $\mathcal{D}(\mathcal{C})$ are the Drinfeld center of \mathcal{C} .
17. Vec_G is the fusion category of G -graded vector spaces, and $\mathcal{D}(G)$ also denotes the Drinfeld center $\mathcal{Z}(\text{Vec}_G)$ of Vec_G .

3 Linear Hopf monads

In this section, we define Hopf monads following Remark 2.7 in [7]. We collect the many different occurrences of HM in the literature and focus on condensation.

3.1 Hopf monads

Definition 3.1. A left Hopf monad on a fusion category \mathcal{C} with associator $a_{X,Y,Z}$ is a quadruple (T, H, η, T_0) , where T is an endofunctor of \mathcal{C} , $H = \{H_{X,Y} \mid X, Y \in \mathcal{C}\}$ is a natural isomorphism from $T(- \otimes T(-))$ to $T(-) \otimes T(-)$, $\eta = \{\eta_X \mid X \in \mathcal{C}\}$ is a natural transformation from $\text{Id}_{\mathcal{C}}$ to T , and $T_0 : T(\mathbf{1}) \rightarrow \mathbf{1}$ is a morphism, such that the following equations hold:

- Compatibility condition between H and η, T_0 :

$$\begin{aligned} H_{X,Y} \eta_{X \otimes T(Y)} &= \eta_X \otimes \text{id}_{T(Y)}, & T_0 \eta_{\mathbf{1}} &= \text{id}_{\mathbf{1}}, \\ (\text{id}_{T(X)} \otimes T_0) H_{X,\mathbf{1}} &= T(\text{id}_X \otimes T_0), & (T_0 \otimes \text{id}_{T(X)}) H_{\mathbf{1},X} T(\eta_X) &= \text{id}_{T(X)}. \end{aligned} \tag{3.1}$$

- The heptagon equation²⁾,

$$\begin{aligned} (\text{id}_{T(X)} \otimes H_{Y,Z}) H_{X,Y \otimes Z} \\ = a_{T(X),T(Y),T(Z)} (H_{X,Y} \otimes T(Z)) H_{X \otimes T(Y),Z} a_{X,T(Y),T(Z)}^{-1} T(\text{id}_X \otimes H_{Y,Z}). \end{aligned} \tag{3.2}$$

From the fusion operator one can derive the multiplication $\mu : T^2 \rightarrow T$ as follows:

$$(T_0 \otimes \text{id}_{T(X)}) H_{\mathbf{1},X} = \mu_X.$$

Furthermore, the comonoidal structure of T is given by

$$T_2(X, Y) = H_{X,Y} T(\text{id}_X \otimes \eta_Y) : T(X \otimes Y) \rightarrow T(X) \otimes T(Y).$$

We shall call the operator H the left fusion operator $H_{X,Y}^l$, and the right fusion operator to be $H_{X,Y}^r$ which operates on $T(T(-) \otimes -)$ instead. They satisfy similar equations as illustrated in [7]. A right Hopf monad has a definition similar to the above using $H_{X,Y}^r$. We shall always assume an HM T to be a left and right HM.

Definition 3.2. A module (M, r) of T is an object M of \mathcal{C} with $r : T(M) \rightarrow M$ a morphism of \mathcal{C} such that

$$rT(r) = r\mu_M, \quad r\eta_M = \text{id}_M.$$

The category of T -modules is denoted by \mathcal{C}^T and its morphisms are \mathcal{C} morphisms preserving the T -module structure.

²⁾ If \mathcal{C} is strict, then two of the maps appearing in the heptagon equation become the identity map, and hence the equation is reduced to a ‘pentagon’ equation, which was first introduced in [7] and is not to be confused with the pentagon equation arising from the associator of a monoidal category or the heptagon equations in the G -crossed braiding.

The HMs we shall work with are always assumed to be semisimple linear and as pointed out in Introduction, for such HMs, the module category is fusion [9].

Two important functors related to T are the forgetful functor $F : \mathcal{C}^T \rightarrow \mathcal{C}$ which forgets the module structure, and its left adjoint $G : M \rightarrow (T(M), \mu_M)$ which gives $T = FG$. This illustrates why every Hopf monad is an adjunction [9, Remark 3.16].

A definition that will be needed to tie the condensable algebra topic to HMs is the following definition.

Definition 3.3. A Hopf comonad $T : \mathcal{C} \rightarrow \mathcal{C}$ is a bicomonad, i.e., a monoidal comonad with a natural morphism called counit $\epsilon : T \rightarrow 1_{\mathcal{C}}$, a unit morphism of objects $T_0 : \mathbf{1} \rightarrow T(\mathbf{1})$, plus a left and right fusion operator satisfying the same axioms of HMs.

Just like HMs, our Hopf comonads will always be assumed to be semisimple linear.

One way to picture a Hopf comonad is to imagine all structural morphisms and axioms of HMs with arrows reversed. A Hopf comonad can also be recovered by taking the *right* adjoint of a tensor functor. Furthermore, the comodules of a Hopf comonad, denoted by \mathcal{C}^{co-T} , form a fusion category.

It can be easily observed that HMs like $T_{\mathcal{H}}(-) = \mathcal{H} \otimes -$ are also Hopf comonads. Therefore, it is useful to have the following definition.

Definition 3.4. An HM T that is also a Hopf comonad is called a self-dual HM.

For a self-dual HM, \mathcal{C}^T and \mathcal{C}^{co-T} will denote the module and comodule categories, respectively.

3.2 Examples of HMs

3.2.1 HMs from group actions

Consider a G -symmetry of a finite group G on \mathcal{C} defined by a strong monoidal functor $T_- : \underline{G} \rightarrow \underline{\text{Aut}}_{\otimes}(\mathcal{C})$. One can form the operator $T_G = \bigoplus_{g \in G} T_g$ where T_g are the symmetries. This is indeed an HM [8, Theorem 4.21]. To see why, the counit T_0 and unit η and the fusion operator H are the analog of the same operators which shows that $\mathbb{C}[G]$ is a Hopf algebra. To define the fusion operator we can use our knowledge of the coproduct

$$T_2 : \bigoplus T_g(X \otimes Y) \rightarrow \bigoplus T_g(X) \otimes T_g(Y),$$

which is derived from the fact that T_g are (strong monoidal) automorphisms, and the definition of the fusion operator in [7, Introduction] as

$$H_{X,Y}^l = (\text{id}_{T(X)} \otimes \mu_Y) T_2(X, T(Y))$$

and similarly for the right $H_{X,Y}^r$. Straightforward calculation shows that both are invertible. This proves that T_G is an HM.

3.2.2 HMs from categorical Hopf algebras

The *action* of a Hopf algebra was used as a venue to generalize its notion to all monoidal categories. This provides the most basic general class of HMs [9]; a Hopf algebra \mathcal{H} inside \mathcal{C} gives an HM $T_{\mathcal{H}}(-) = \mathcal{H} \otimes -$. These are Hopf algebra symmetries. Notice the Hopf algebra needs a braiding for its definition, and therefore a braiding structure for the relevant category is always assumed.

3.2.3 HMs from tensor functors

Tensor functors are extra structures on functors between fusion categories. A tensor functor $F : \mathcal{D} \rightarrow \mathcal{C}$ between fusion categories has adjoints (see, e.g., [8, Subsection 1.3]) and its left adjoint G provides an HM $T = FG$ and an equivalence $\mathcal{D} \cong \mathcal{C}^T$. Conversely, any HM T has a (forgetful) tensor functor $U : \mathcal{C}^T \rightarrow \mathcal{C}$ with the left adjoint being $X \rightarrow (T(X), \mu_X)$.

3.2.4 HMs from condensation

We first define a condensable algebra. To do so, recall an algebra \mathcal{A} of a braided fusion category \mathcal{B} with multiplication $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is

- commutative if $m \circ c_{\mathcal{A},\mathcal{A}} = m$ where $c_{\mathcal{A},\mathcal{A}}$ is the braiding,

- connected if $\text{Hom}(\mathbf{1}, \mathcal{A}) \cong \mathbb{C}$,
- separable if m admits a splitting $\zeta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, a morphism of $(\mathcal{A}, \mathcal{A})$ -bimodules.

Definition 3.5. An algebra \mathcal{A} of a braided fusion category \mathcal{B} is condensable if it is a connected etale (commutative and separable) algebra.

A condensable algebra \mathcal{A} (physically) will also be referred to as normal (mathematically).

For the HM $T_{\mathcal{H}}(-) = \mathcal{H} \otimes -$ given by a Hopf algebra, the forgetful functor is the tensor functor and we consider its left adjoint. The case of condensable algebras (which are not Hopf algebras in general) is opposite and we refer to [11, Definitions 3.11 and 3.12] for the details. The condensation functor $D_{\mathcal{A}} : \mathcal{B} \rightarrow \mathcal{B}_{\mathcal{A}}$ is the tensor functor, and its *right* adjoint is the forgetful functor $E_{\mathcal{A}} : \mathcal{B}_{\mathcal{A}} \rightarrow \mathcal{B}$. The composition $T_{\mathcal{A}} = D_{\mathcal{A}} \circ E_{\mathcal{A}} : \mathcal{B}_{\mathcal{A}} \rightarrow \mathcal{B}_{\mathcal{A}}$ is a Hopf comonad and the equivalence $\mathcal{B} \cong \mathcal{B}_{\mathcal{A}}^{\text{co-}T_{\mathcal{A}}}$ holds.

A simple object X of $\mathcal{B}_{\mathcal{A}}$ is deconfined if all the simple objects of $E_{\mathcal{A}}(X)$ all have the same topological twist. Otherwise it is confined and called a defect.

In order to derive the maps $D_{\mathcal{A}}$ and $E_{\mathcal{A}}$, the Frobenius reciprocity property is used

$$\text{Hom}_{\mathcal{B}}(X, \mathcal{A} \otimes Y) = \text{Hom}_{\mathcal{B}_{\mathcal{A}}}(X, Y). \tag{3.3}$$

Notice that the objects of the two categories are actually the same although the simple objects are of course not. We should note that the above version of Frobenius reciprocity is true for the cases considered in this paper. Otherwise, instead of $\mathcal{B}_{\mathcal{A}}$ there should be $\widetilde{\mathcal{B}}_{\mathcal{A}}$ which is a pre-quotient category which needs to go through an idempotent completion process to give us $\mathcal{B}_{\mathcal{A}}$ (see [11, Definitions 3.11 and 3.12]). When there are no non-trivial splitting idempotents in $\widetilde{\mathcal{B}}_{\mathcal{A}}$, then $\widetilde{\mathcal{B}}_{\mathcal{A}} = \mathcal{B}_{\mathcal{A}}$. The Frobenius reciprocity provides enough constraints to derive completely the maps $D_{\mathcal{A}}, E_{\mathcal{A}}$ for the condensation examples in Subsection 5.2.

3.3 Double of HMs

The double of T is defined using the coend

$$Z_T(x) = \bigoplus_i {}^*T(x_i) \otimes x \otimes x_i$$

as $D_T = Z_T \circ T$, where the summation is finite as we deal with UFCs ([9, Subsection 9.2]). The important example is $T = 1_{\mathcal{C}}$ which gives the double of \mathcal{C} , and in general ([9, Subsection 9.2]):

$$\mathcal{C}^{D_T} \cong \mathcal{Z}(\mathcal{C}^T),$$

as braided categories. This simple example also shows the power of Hopf monads, where the most simple HM, the identity map, can give rise to the double construction.

3.4 Equivariantization of HM

The equivariantization \mathcal{C}^G of the group action G on \mathcal{C} is known to be equivalent to \mathcal{C}^{T_G} [8, Theorem 4.21]. Therefore, the natural generalization of equivariantization is the concept of T -modules. This also makes sense as these notions are supposed to categorify the notion of fixed point of an action. Therefore, the last step in the gauging process is the process of taking the T -modules.

4 Skeletal definition of Hopf monads

In mathematics, many notions such as a fusion category are equivalence classes. It is often convenient to work with different representatives of the same equivalence class. In the case of a fusion category, two extreme convenient representatives are: a strict version with many objects, and a skeletal version with the least number of objects. The strict version is convenient for general discussions and pictorial calculus, while the skeletal version is easier for concrete calculations. In this section, we provide the skeletal definition of an HM, presented as solutions of polynomial equations. As a warm-up, we start with the skeletal definition of tensor functors.

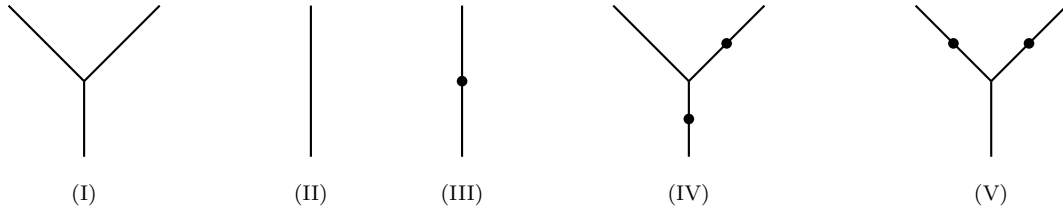


Figure 1 (I) \otimes ; (II) $\text{Id}_{\mathcal{C}}$; (III) T ; (IV) $T \circ \otimes \circ (\text{Id} \times T)$; (V) $\otimes \circ (T \times T)$

4.1 Diagrammatic notations

Let \mathcal{C} be a fusion category and $L(\mathcal{C}) = \{a, b, c, \dots\}$ be a complete set of representatives, i.e., a set that contains a representative for each isomorphism class of simple objects of \mathcal{C} . Let N_{ab}^c be the fusion coefficients,

$$a \otimes b \simeq \bigoplus_{c \in L(\mathcal{C})} N_{ab}^c c.$$

For simplicity, we assume \mathcal{C} to be multiplicity free, i.e., $N_{ab}^c = 0, 1$. A triple (a, b, c) is called admissible if $N_{ab}^c = 1$.

We introduce a diagrammatic way to represent the tensor product functor, an extra functor T , and their Cartesian product/compositions. We first assume T to be an endofunctor of \mathcal{C} . Then T is represented by an integer-valued matrix (T_{ab}) :

$$T(b) = \bigoplus_{a \in L(\mathcal{C})} T_{ab} a, \quad b \in L(\mathcal{C}).$$

Consider binary forests, i.e., a disjoint union of binary trees, with each edge possibly decorated with some number of dots. Every dot on the edge splits that edge into two. (If there are n dots on an edge, then that edge is split into $n + 1$ edges.) The forests have distinguished roots and are aligned so that they grow in the upward direction. Thus roots appear at the bottom and leaves appear at the top. Let \mathcal{G} be a forest as described above. If \mathcal{G} has n leaves and m roots, then it is interpreted as a functor $\mathcal{C}^n \rightarrow \mathcal{C}^m$ according to the following rules.

- A ‘Y’-shape tree is interpreted as the tensor product functor $\otimes: \mathcal{C}^2 \rightarrow \mathcal{C}$; a verticle segment as the identity functor $\text{Id}: \mathcal{C} \rightarrow \mathcal{C}$; a verticle segment with a dot in it as $T: \mathcal{C} \rightarrow \mathcal{C}$.

- Horizontal juxtaposition of two forests corresponds to the Cartesian product of the respective functors.

- Vertical concatenation of two forests corresponds to the composition of the respective functors. Explicitly, if the roots of \mathcal{G}_1 coincide with the leaves of \mathcal{G}_2 , then the concatenated forest $\mathcal{G}_1 \# \mathcal{G}_2$ is interpreted as the composition of the functor corresponding to \mathcal{G}_1 and that corresponding to \mathcal{G}_2 .

See Figure 1 for examples of forests and their interpretations as functors. From now on, we will only study trees and we will use the same symbol for a tree and the functor it represents.

Let \mathcal{G} be a binary tree with n leaves representing a functor (also denoted by \mathcal{G}) from \mathcal{C}^n to \mathcal{C} . We label each leaf l_i with a simple object $a_i \in L(\mathcal{C})$. Then $\mathcal{G}(a_1, \dots, a_n)$ is an object in \mathcal{C} which may contain multiple copies of each simple object in $L(\mathcal{C})$. To specify a particular copy, we further label all other edges (including the one attached to the root) with simple objects in $L(\mathcal{C})$ and label all dots with positive integers so that the following two conditions hold:

1. at each trivalent vertex, if the three edges (pointing towards to the northwestern direction, northeastern direction, southern direction) are respectively labelled by a, b, c , then (a, b, c) is admissible;
2. at each dot, if the edge above (respectively, below) the dot is labelled by b (respectively, a) and if the dot is labelled by α , then $1 \leq \alpha \leq T_{ab}$.

See Figure 2 for an illustration of labels of trees.

If the label for the edge attached to the root is b , then a labelling of \mathcal{G} determines a specific copy of b inside $\mathcal{G}(a_1, \dots, a_n)$, or equivalently, a basis element in the space $\text{Hom}(b, \mathcal{G}(a_1, \dots, a_n))$. By ranging all

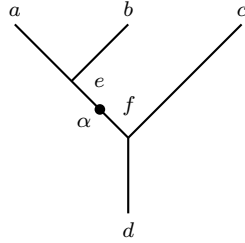


Figure 2 A labelling of a tree

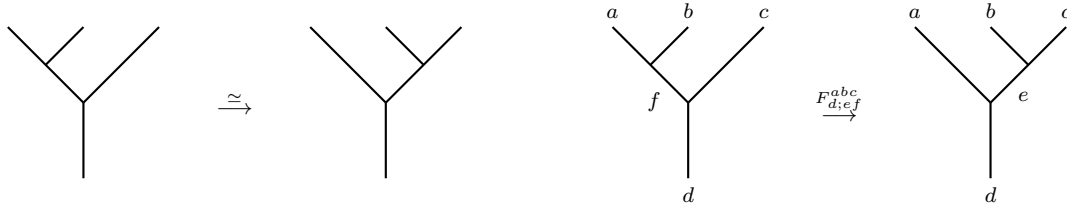


Figure 3 (Left) the associator natural isomorphism; (Right) the matrix elements of the associator natural isomorphism in the basis given by labels of internal edges

labels of internal edges, we enumerate all copies of b , or equivalently, a basis in $\text{Hom}(b, \mathcal{G}(a_1, \dots, a_n))$. We will use these two interpretations interchangeably.

Remark 4.1. Actually, the basis element in $\text{Hom}(b, \mathcal{G}(a_1, \dots, a_n))$ corresponding to a labelling can be seen directly from the labelled tree. To achieve this, for each admissible (a, b, c) , fix a morphism $B_c^{ab} \in \text{Hom}(c, a \otimes b)$ and for each $a, b \in L(mcC)$, $1 \leq \alpha \leq T_{ab}$, fix an embedding $I_a^b(\alpha): a \rightarrow T(b)$ so that the image of $I_a^b(\alpha)$ is the α -th copy of a inside $T(b)$. Then, divide the tree into several layers such that within each layer the tree is a horizontal juxtaposition of at most three types of graphs: ‘Y’-shape graphs, verticle lines, and verticle lines with a dot. Each ‘Y’-shape graph with labels (a, b, c) is interpreted as B_c^{ab} ; each verticle line with a label a is interpreted as $\text{id}_a \in \text{Hom}(a, a)$; each verticle line with a dot whose labels are $(a, b; \alpha)$ with b being the label of the edge above the dot is interpreted as $I_a^b(\alpha)$. Now, for the i -th layer, assume the morphisms we obtained by the above rules are f_{i1}, \dots, f_{in} and assume the part of the tree below the i -th layer represents the functor F_i , and then the i -th layer is assigned the morphism $F_i(f_{i1}, \dots, f_{in})$. Lastly, the morphism corresponding to the whole tree is obtained as the compositions of the morphisms assigned to each layer, from bottom to top. For example, the labelled tree in Figure 2 represents the morphism:

$$d \xrightarrow{B_d^{fc}} f \otimes c \xrightarrow{I_f^e(\alpha) \otimes \text{id}_c} T e \otimes c \xrightarrow{T(B_e^{ab}) \otimes \text{id}_c} T(a \otimes b) \otimes c. \tag{4.1}$$

To summarize, a binary tree \mathcal{G} represents the functor \mathcal{G} ; a binary tree with leaves labelled by a_i and root labelled by b represents the space $\text{Hom}(b, \mathcal{G}(a_1, \dots, a_n))$; a binary tree with labels on each edge represents a particular copy of b in $\mathcal{G}(a_1, \dots, a_n)$ or a particular basis element in $\text{Hom}(b, \mathcal{G}(a_1, \dots, a_n))$.

Let \mathcal{G}_1 and \mathcal{G}_2 be two binary trees with the same labels $\{a_i\}$ on leaves and b on their root. If Φ is a natural transformation from \mathcal{G}_1 to \mathcal{G}_2 , then with these labels Φ can be written as a matrix $(\Phi_b^{\{a_i\}})_{\{c_i\}, \{d_i\}}$, where $\{c_i\}$ ranges over all labels of internal edges of \mathcal{G}_2 and $\{d_i\}$ ranges those of \mathcal{G}_1 . This is how we will represent natural transformations in the skeletal definition of tensor functors and Hopf monads below.

As an example, the associator isomorphism for the tensor product and its matrix elements, the so called F -matrix, are represented as in Figure 3.

By considering a more restricted class of binary trees, we can relax the condition of T to also allow T to be a functor from \mathcal{C} to another fusion category \mathcal{D} . Basically, a dot is still interpreted as T and a ‘Y’-shape graph is interpreted either as the tensor product in \mathcal{C} or as that in \mathcal{D} depending on the specific configuration. Labels of edges will also be simple objects from either $L(\mathcal{C})$ or $L(\mathcal{D})$. The horizontal juxtaposition/vertical concatenation rules remain the same as before. We will not give an explicit char-

acterization for such class of trees since all such trees we will be using in the next subsection are simple enough so that the readers can easily verify that those allow T to take a target different from \mathcal{C} .

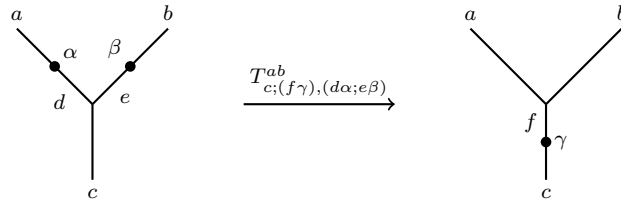


Figure 4 The matrix elements of T_c^{ab}

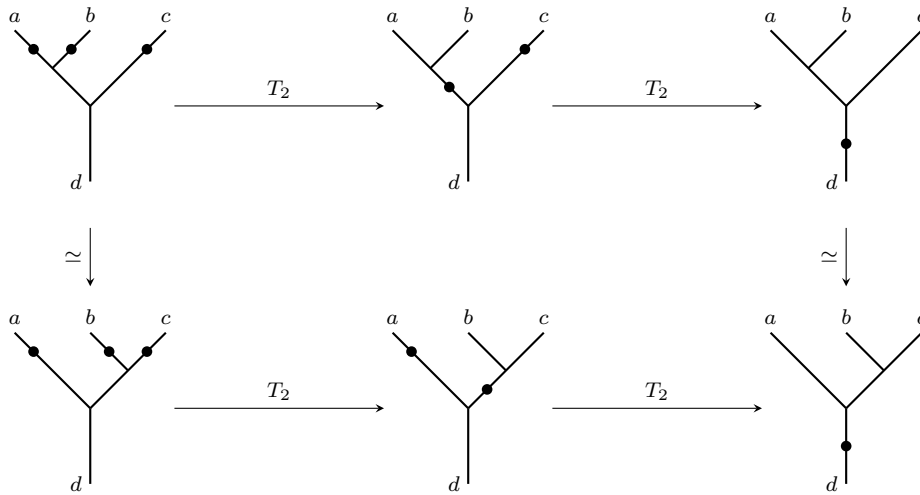


Figure 5 The hexagon equation for T_2

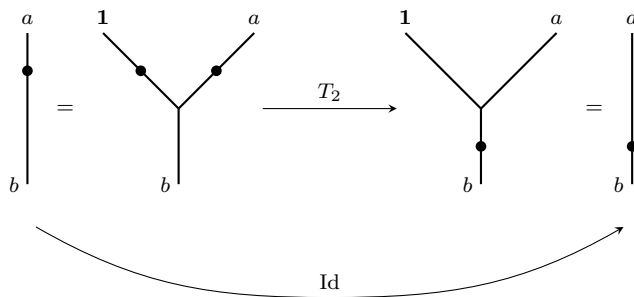


Figure 6 The first triangle equation for T_2

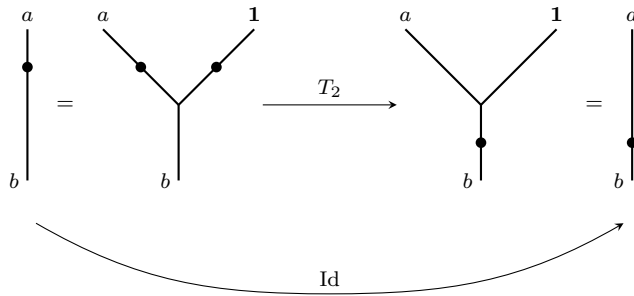


Figure 7 The second triangle equation for T_2

4.2 Tensor functors

Since the tensor functor is a fundamental notion in our theory, we provide a skeletal definition in this subsection. We will use diagrammatic notations introduced in the previous subsection.

Let \mathcal{C} and \mathcal{D} be multiplicity-free fusion categories and without loss of generality, assume that the left/right unit isomorphisms $\mathbf{1} \otimes X \simeq X \simeq X \otimes \mathbf{1}$ in \mathcal{C} and \mathcal{D} are all the identity maps. We will use English letters to denote simple objects and Greek letters to denote integers.

Recall that a tensor functor $(T, T_2, T_0) : \mathcal{C} \rightarrow \mathcal{D}$ is a linear strong monoidal functor that is faithful and exact. Such a functor determines, and is determined by, the data $(T_{ab}, T_{c;(f\gamma), (d\alpha; e\beta)}^{ab})$ satisfying (4.5)–(4.8).

For each $b \in L(\mathcal{C})$,

$$T(b) = \bigoplus_{a \in L(\mathcal{D})} T_{ab} a, \quad T_{ab} \in \mathbb{N}. \tag{4.2}$$

For each $a, b \in L(\mathcal{C})$, there exists a natural isomorphism,

$$T_2(a, b) : T(a) \otimes T(b) \xrightarrow{\sim} T(a \otimes b). \tag{4.3}$$

The isomorphism $T_2(a, b)$ determines a family of invertible matrices $\{T_c^{ab} : a, b \in L(\mathcal{C}), c \in L(\mathcal{D})\}$, where the matrix elements of T_c^{ab} are given by

$$\{T_{c;(f\gamma), (d\alpha; e\beta)}^{ab} : f \in L(\mathcal{C}), d, e \in L(\mathcal{D}), 1 \leq \alpha \leq T_{da}, 1 \leq \beta \leq T_{eb}, 1 \leq \gamma \leq T_{cf}\}, \tag{4.4}$$

or diagrammatically shown in Figure 4. In particular, T_c^{ab} being an isomorphism implies that the number of rows and the number of columns coincide,

$$\sum_{d, e} T_{da} T_{eb} N_{de}^c = \sum_f N_{ab}^f T_{cf}, \quad \forall a, b \in L(\mathcal{C}), \quad c \in L(\mathcal{D}). \tag{4.5}$$

Without loss of generality, we assume $T_0 : \mathbf{1} \rightarrow T(\mathbf{1})$ is the identity map. Thus,

$$T_{a\mathbf{1}} = \delta_{a, \mathbf{1}}. \tag{4.6}$$

The axioms T_2 needs to satisfy are shown in Figures 5–7, where we omitted labels of internal edges. Correspondingly, we obtain the following equations:

$$\begin{aligned} \sum_{l, \lambda} T_{h; (i\omega), (e\alpha; f\beta)}^{ab} T_{d; (j\theta), (h\omega; g\gamma)}^{ic} F_{j; ki}^{abc} &= \sum_{i, \omega} F_{d; lh}^{efg} T_{l; (k\lambda), (f\beta; g\gamma)}^{bc} T_{d; (j\theta), (e\alpha; l\lambda)}^{ak}, \\ \forall a, b, c \in L(\mathcal{C}), \quad d, e, f, g, h, j, k \in L(\mathcal{D}), \\ 1 \leq \alpha \leq T_{ea}, \quad 1 \leq \beta \leq T_{fb}, \quad 1 \leq \gamma \leq T_{gc}, \quad 1 \leq \theta \leq T_{dj}, \end{aligned} \tag{4.7}$$

$$T_{b; (a\beta), (\mathbf{1}\mathbf{1}; b\alpha)}^{\mathbf{1}a} = T_{b; (a\beta), (b\alpha; \mathbf{1}\mathbf{1})}^{\mathbf{1}a} = \delta_{\alpha, \beta}, \quad \forall a \in L(\mathcal{C}), \quad b \in L(\mathcal{D}), \quad 1 \leq \alpha \leq T_{ba}. \tag{4.8}$$

Note that in Equation (4.7), the F -matrix on the left-hand side is computed in \mathcal{C} while the F -matrix on the right-hand side is computed in \mathcal{D} .

4.3 Hopf monads

We give a skeletal definition of a left Hopf monad. The corresponding definition for a right Hopf monad can be carried out analogously. Again let \mathcal{C} be a rank= n multiplicity-free fusion category with a complete set of representatives $L = L(\mathcal{C}) = \{a, b, c, \dots\}$.

Let (T, H, T_0, η) be a left Hopf monad on \mathcal{C} according to Definition 3.1. T is a functor and is determined by the $n \times n$ integer-valued matrix (T_{ab}) :

$$T(b) = \bigoplus_{a \in L} T_{ab} a, \quad b \in L. \tag{4.9}$$

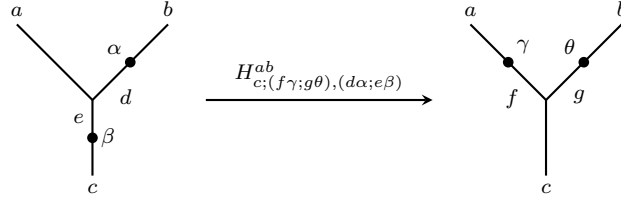


Figure 8 The matrix elements of H_c^{ab}

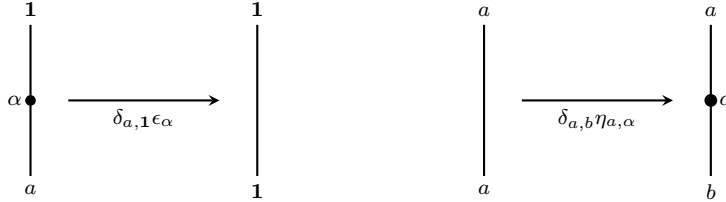


Figure 9 The matrix elements of T_0 (left) and η (right)

H is a natural isomorphism from $T \circ \otimes \circ (\text{Id} \times T)$ to $\otimes \circ (T \times T)$. The diagrammatical representations for the above two functors are represented in Figures 1(IV) and 1(V), respectively. Hence H is a collection of invertible matrices $\{H_c^{ab} : a, b, c \in L\}$, where the matrix elements of H_c^{ab} are given by

$$\{H_{c;(f\gamma;g\theta),(d\alpha;e\beta)}^{ab} : d, e, f, g \in L, 1 \leq \alpha \leq T_{db}, 1 \leq \beta \leq T_{ce}, 1 \leq \gamma \leq T_{fa}, 1 \leq \theta \leq T_{gb}\}, \tag{4.10}$$

or diagrammatically illustrated in Figure 8. In particular, the number of rows and the number of columns of H_c^{ab} must be equal:

$$\sum_{f,g \in L} T_{fa} T_{gb} N_{fg}^c = \sum_{d,e \in L} T_{db} T_{ce} N_{ad}^e. \tag{4.11}$$

The matrix elements for $T_0 : T(\mathbf{1}) \rightarrow \mathbf{1}$ are determined by a vector

$$\{\epsilon_\alpha : 1 \leq \alpha \leq T_{11}\}, \tag{4.12}$$

and the natural transformation $\eta : 1_C \rightarrow T$ gives a family of vectors $\{\eta_a : a \in L\}$ where η_a is

$$\{\eta_{a,\alpha} : 1 \leq \alpha \leq T_{aa}\}. \tag{4.13}$$

See Figure 9 for a diagrammatic definition of the vectors ϵ and η_a .

The heptagon axiom for H is the equality of two paths of natural transformations from the functor

$$T \circ \otimes \circ (\text{Id} \times T \circ \otimes) \circ (\text{Id} \times \text{Id} \times T)$$

to $\otimes \circ (\text{Id} \times \otimes) \circ (T \times T \times T)$. The diagrammatic representation of the axiom is presented in Figure 10, from which we derive the heptagon equation:

$$\begin{aligned} & \sum_{\theta} H_{d;(g'\beta';h'\theta),(g\beta;h\gamma)}^{af} H_{h';(e'\alpha';f'\gamma'),(e\alpha;fw)}^{bc} \\ &= \sum_{\tilde{e}, \tilde{f}, \tilde{g}, \tilde{f}\tilde{\alpha}, \tilde{\beta}, \tilde{\beta}} H_{g;(\tilde{e}\tilde{\alpha};\tilde{f}\tilde{\beta}), (e\alpha;f\beta)}^{bc} (F_h^{a\tilde{e}\tilde{f}})^{-1} \tilde{g}\tilde{g} H_{d;(\tilde{f}\tilde{\beta};f'\gamma'), (\tilde{f}\tilde{\beta};h\gamma)}^{\tilde{g}c} H_{f;(g'\beta';e'\alpha'), (\tilde{e}\tilde{\alpha};\tilde{g}\tilde{\beta})}^{ab} F_{d;h'\tilde{f}}^{g'e'f'}. \end{aligned} \tag{4.14}$$

Conditions in Equation (3.1) can be represented in a similar way by diagrams. As an example, we illustrate the diagrams for the first two equalities in Figures 11 and 12, respectively, and leave the rest as an exercise. The four equalities in Equation (3.1) are written, in terms of matrix elements, in the following equations:

$$\begin{aligned} & \sum_{\beta} \eta_{c,\beta} H_{c;(e\gamma;f\theta),(d\alpha;c\beta)}^{ab} = \delta_{f,d} \delta_{e,a} \delta_{\theta,\alpha} \eta_{a,\gamma}, \quad \sum_{\alpha} \eta_{1,\alpha} \epsilon_{\alpha} = 1, \\ & \sum_{\theta} H_{b;(b\gamma;1\theta),(c\alpha;d\beta)}^{a1} \epsilon_{\theta} = \delta_{c,1} \delta_{d,a} \delta_{\gamma,\beta} \epsilon_{\alpha}, \quad \sum_{\beta,\gamma} H_{b;(1\gamma;b\theta),(a\beta;a\alpha)}^{1a} \eta_{a\beta} \epsilon_{\gamma} = \delta_{\theta,\alpha}. \end{aligned} \tag{4.15}$$

In summary, a left Hopf monad is defined by the data $\{T_{ab}, H_{c;(f\gamma;g\theta), (d\alpha;e\beta)}, \epsilon_\alpha, \eta_{a,\alpha}\}$ satisfying Equations (4.11), (4.14) and (4.15).

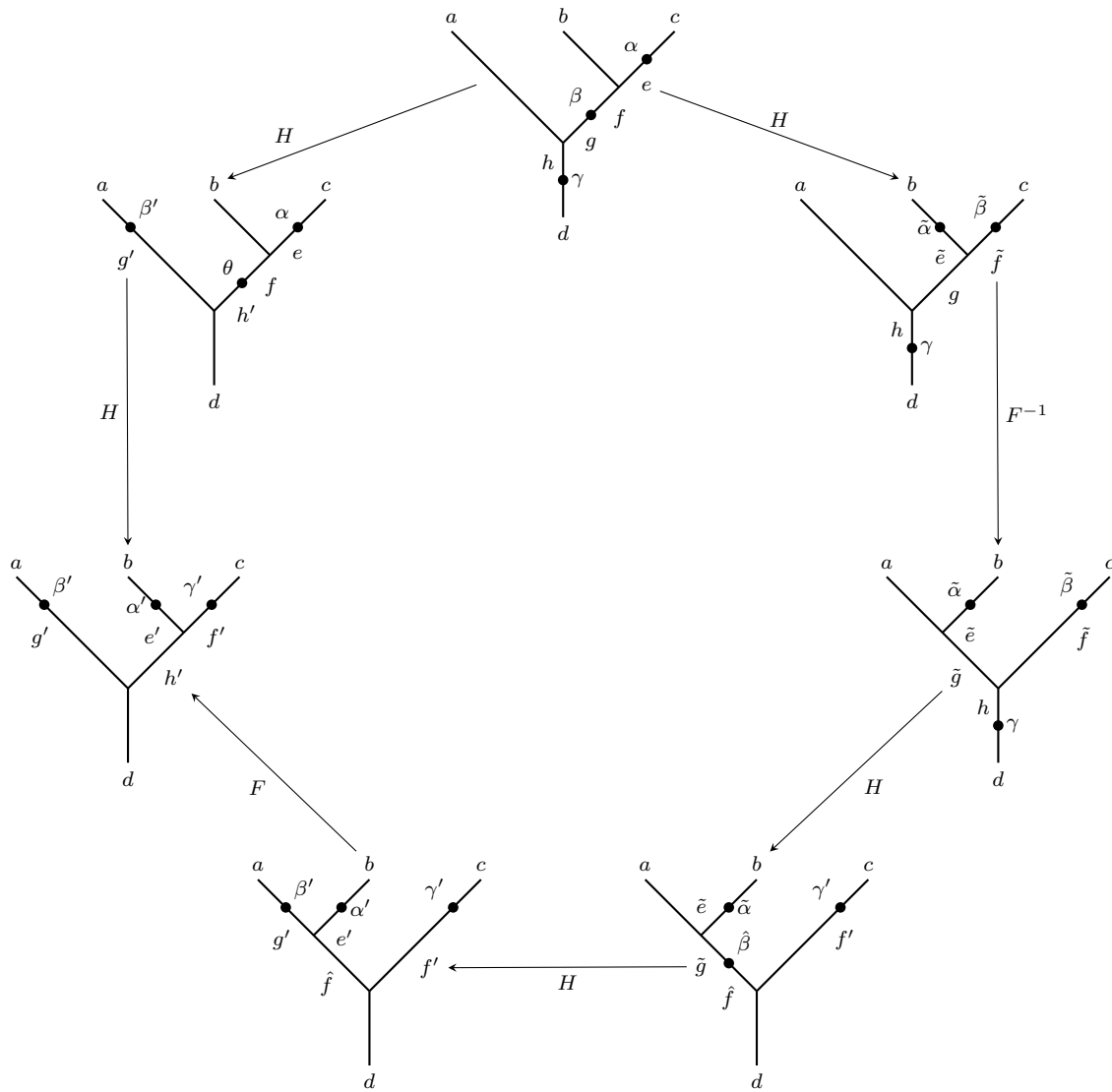


Figure 10 Heptagon equation for H

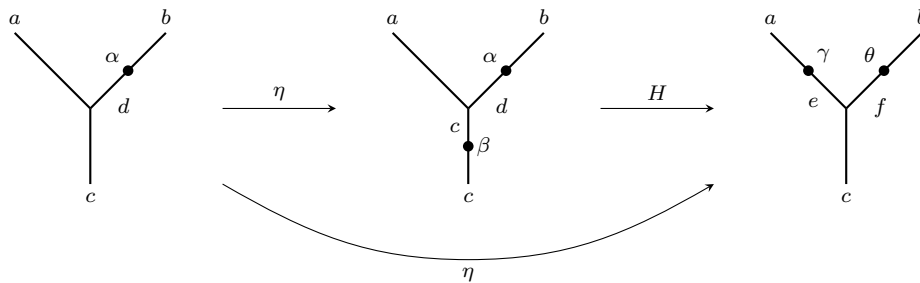


Figure 11 The first equality in Equation (3.1)

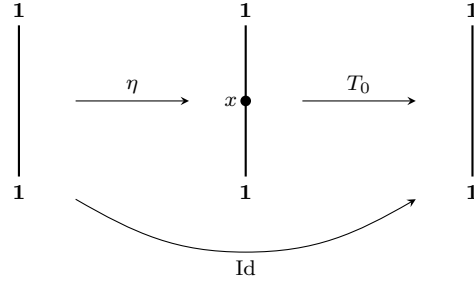


Figure 12 The second equality in Equation (3.1)

5 Classification of Hopf monads

In this section, we consider HMs in $Vec_{\mathbb{Z}_p}$ as well as those from condensations. For $p = 1$, the classification of HMs is reduced to the classification of Hopf algebras over \mathbb{C} , and this most trivial case shows the difficulty of classifying all HMs. Hence, instead of classifying all HMs, we attempt to characterize them and find many interesting examples. At the same time, we will focus on some examples and describe their module categories, where the classification of indecomposable module categories over Vec_G (see, e.g., [16, Subsection 2.7]) shall be used. Every indecomposable module category over Vec_G is essentially determined by

- a group algebra $\mathbb{C}[K]$ for a subgroup $K \subset G$,
- a 2-cocycle ψ on K determining the multiplication in $\mathbb{C}[K]$.

The module category associated with (K, ψ) has equivalence classes of simple objects corresponding to the left cosets G/K and the Vec_G action is the obvious one: $g.[aK] = [gaK]$.

5.1 Characterization of HMs on $Vec_{\mathbb{Z}_p}$ for p prime

5.1.1 Vec

An HM T is determined by its action on the unit object and $T(\mathbf{1}) = \mathcal{H} \Rightarrow T(X) = \mathcal{H} \otimes X$. It is then easy to observe that the Hopf monad axioms, with all objects present replaced by the unit, imply that \mathcal{H} is a Hopf algebra. Therefore, all possible HMs on Vec are determined by an *extension* of the Hopf algebra structure. If one assumes the HM structural morphisms (μ, η, \dots) to be additive as well, then the T structure is entirely determined by \mathcal{H} as all objects in Vec are sums of copies of unit, i.e., any T is of the form $\mathcal{H} \otimes -$.

Remark 5.1. As explained above, in case an HM is of the form $T(-) = \mathcal{H} \otimes -$ at the object level, it does not necessarily imply that the HM structure is entirely determined by the structure of \mathcal{H} . From now on, all HMs of the form $T(-) = \mathcal{H} \otimes -$ for a Hopf algebra \mathcal{H} are assumed to have their structure inherited from that of \mathcal{H} .

5.1.2 $Vec_{\mathbb{Z}_2} = \{\mathbf{1}, e\}$

In general for Vec_G , it can be observed that a Hopf algebra is simply a G -graded Hopf algebra $\mathcal{H} = \bigoplus \mathcal{H}_g$. So for $G = \mathbb{Z}_2$, a Hopf algebra is an extension of the structure on a super-algebra. We cannot characterize any further Hopf algebras, as its most trivial case contains all Hopf algebras over Vec (when $\mathcal{H}_g = 0$ for $g \neq 1$). But we will examine the case of $2 + e$ which is applied in an example of condensation. Note that here e denotes the non-trivial simple object in $Vec_{\mathbb{Z}_2}$.

Super-algebra structures on $2 + e$ have been classified in [1]. We wish to extend those to a Hopf algebra structure. Notice that this is different from the notion of a *Hopf superalgebra* in [1] which has a fermionic braiding, while the braiding here in Vec_G is trivial. Mathematica computations show that there is a unique Hopf algebra structure on $\mathcal{H} = 2 + e = \mathbb{C}^2 \oplus \mathbb{C}$. As a super-algebra, it is identified with the \mathbb{Z}_2 -graded algebra $\mathbb{C}[x, y]/\langle x^2 - x, y^2 - x, xy - y \rangle$, where 1 and x have grading 0 and y has grading 1.

With this identification, other structural maps are given as follows:

$$\Delta(x) = 1 \otimes x + x \otimes 1 - \frac{3}{2}x \otimes x + \frac{1}{2}y \otimes y, \tag{5.1}$$

$$\Delta(y) = 1 \otimes y - \frac{3}{2}x \otimes y + y \otimes 1 - \frac{3}{2}y \otimes x,$$

$$\epsilon(x) = \epsilon(y) = 0, \tag{5.2}$$

$$S(x) = x, \quad S(y) = -y. \tag{5.3}$$

By definition, \mathcal{H} is commutative. It also has an integral $1 - x$ and $\epsilon(1 - x) = 1$. As a graded algebra, \mathcal{H} decomposes into the direct sum of two indecomposable graded subalgebras,

$$\mathcal{H} = \text{Span}\{1 - x\} \oplus \text{Span}\{x, y\}, \tag{5.4}$$

or abstractly $2 + e = \mathbf{1} + (\mathbf{1} + e)$. The algebra structures on $\mathbf{1}$ and $\mathbf{1} + e$ correspond to the trivial subgroup and \mathbb{Z}_2 , respectively. Therefore, by definition ([28, Definition 9 and Remark 5]), \mathcal{H} is semisimple and there are three irreducible graded modules A, B and C . Indeed, \mathcal{H} decomposes to two algebras $\mathbf{1}$ and $\mathbf{1} + e$, therefore its module category decomposes to the two module categories for the two algebras and we already know that these algebras give indeed semisimple abelian categories as their module category.

Remark 5.2. Here is an observation that will be used repeatedly later. A Hopf algebra \mathcal{H} in \mathcal{B} always has the trivial irreducible module $(\mathbf{1}, \epsilon)$ and moreover for any simple object $h \in \mathcal{B}$, $(h, \epsilon_{\mathcal{H}} \otimes \text{id}_h)$ is also an irreducible module, where $\epsilon_{\mathcal{H}} \otimes \text{id}_h: \mathcal{H} \otimes h \rightarrow h$ is the module map.

A is the trivial module induced by the counit. B is the 1-dimensional space with the non-zero part concentrated on degree 1. The action of \mathcal{H} on B is also given by the counit. C is the 2-dimensional space spanned by x and y , and the action of \mathcal{H} is given by multiplication. Since $1 - x$ is an integral, $\text{Span}\{1 - x\}$ as an \mathcal{H} -module is isomorphic to the trivial irrep A , and hence the regular representation decomposes as $\mathcal{H} = A \oplus C$.

It is direct to see that the three modules $\{A, B, C\}$ form the category $\text{Rep}(S_3)$. For example, the decomposition $C \otimes C = A + B + C$ is given as follows:

$$\begin{aligned} A &= \text{Span}\{x \otimes x - y \otimes y\}, & B &= \text{Span}\{x \otimes y - y \otimes x\}, \\ C &= \text{Span}\{x \otimes x + y \otimes y, -x \otimes y - y \otimes x\}, \end{aligned} \tag{5.5}$$

where the first term in $C = \text{Span}\{\dots\}$ corresponds to x in C and the second corresponds to y .

Remark 5.3. The Hopf structure on $2 + e$ is an explicit example of a categorical Hopf algebra that has no analogue in Vec as the only semisimple 3-dimensional Hopf algebra is the group algebra $\mathbb{C}[\mathbb{Z}_3]$.

5.1.3 $\text{Vec}_{\mathbb{Z}_3} = \{\mathbf{1}, e, e^2\}$

We would like to see how much the same pattern repeats itself. Inspired by the previous case where the Hopf algebra decomposed as an algebra, one can consider the direct sum of the two algebras $\mathcal{H} = \mathbf{1} \oplus \text{Vec}_{\mathbb{Z}_3}$, where by abusing notation we use $\text{Vec}_{\mathbb{Z}_3}$ to denote the object $\mathbf{1} \oplus e \oplus e^2$. As a graded algebra, \mathcal{H} is identified with $\mathbb{C}[y_1]/\langle y_1^4 - y_1 \rangle$ where y_1 has degree 1. Hence a basis for \mathcal{H} is given by $\{1, y_1^3, y_1, y_1^2\}$. To do computations with Mathematica, one can use the fact that $\Delta(y_1)^i = \Delta(y_1^i)$. This greatly reduces the number of parameters in the computation. With this trick, all possible Hopf algebra extensions for $\mathbf{1} \oplus \text{Vec}_{\mathbb{Z}_3}$ can be recovered. In fact, it turns out that there is only one, whose structure maps are as follows. The counit satisfies $\epsilon(1) = 1$, $\epsilon(y_1^i) = 0$, $i = 1, 2, 3$. The antipode S is the identity map unlike the previous case, and the comultiplication is given as follows:

$$\begin{aligned} \Delta(1) &= 1 \otimes 1, \\ \Delta(y_1^3) &= 1 \otimes y_1^3 + y_1^3 \otimes 1 - \frac{4}{3}y_1^3 \otimes y_1^3 - \frac{1}{3}(y_1 \otimes y_1^2 + y_1^2 \otimes y_1), \\ \Delta(y_1) &= 1 \otimes y_1 + y_1 \otimes 1 + \frac{2}{3}y_1^2 \otimes y_1^2 - \frac{4}{3}(y_1 \otimes y_1^3 + y_1^3 \otimes y_1), \\ \Delta(y_1^2) &= 1 \otimes y_1^2 + y_1^2 \otimes 1 + \frac{2}{3}y_1 \otimes y_1 - \frac{4}{3}(y_1^2 \otimes y_1^3 + y_1^3 \otimes y_1^2). \end{aligned} \tag{5.6}$$

Similar to the previous case, the decomposition as subalgebras $\mathcal{H} = \text{Span}\{1 - y_1^3\} \oplus \text{Span}\{y_1^3, y_1, y_1^2\}$ corresponds abstractly to the decomposition $\mathcal{H} = \mathbf{1} \oplus \text{Vec}_{\mathbb{Z}_3}$.

The irreducible modules of \mathcal{H} are given by its algebra decomposition. The algebra $\text{Span}\{1 - y_1^3\}$ gives three modules from the action of counit denoted by $\{y_1^3, y_1, y_1^2\}$, and the second algebra $\text{Span}\{y_1^3, y_1, y_1^2\}$ gives itself as a module, called ρ , with the action given by multiplication. The module category is therefore $\{y_1^3, y_1, y_1^2, \rho\}$. As $\{y_1^3, y_1, y_1^2\}$ forms a $\text{Vec}_{\mathbb{Z}_3}$ as a fusion category, it remains to understand how ρ fuses with itself and other simple modules. This is all done by direct calculations and the fusion rules turn out to be a generalization of the previous case, giving the near-group category with the so-called multiplicity $m = |\mathbb{Z}_3| - 1 = 2$:

$$\rho \otimes \rho = y_1^3 \oplus y_1 \oplus y_1^2 \oplus 2\rho, \tag{5.7}$$

$$\rho \otimes \alpha = \rho, \quad \forall \alpha \in \{y_1^3, y_1, y_1^2\}. \tag{5.8}$$

This is also noted [23, Theorem 1.1] to be $\text{Rep}(A_4)$, the representation category of the alternating group A_4 .

For a general prime p , we can also consider the algebra $\mathbf{1} \oplus \text{Vec}_{\mathbb{Z}_p}$ in $\text{Vec}_{\mathbb{Z}_p}$ where the first $\text{Vec}_{\mathbb{Z}_p}$ denotes the direct sum of all simple objects, each with multiplicity one. The cases for $p = 2$ and $p = 3$ motivate the conjecture that the algebra $\mathbf{1} \oplus \text{Vec}_{\mathbb{Z}_p}$ has a Hopf algebra structure with representation category the near group category given by $G = \mathbb{Z}_p$ and multiplicity $m = |G| - 1$. However, a classification by [21, Theorem 5.1] shows that can only happen if $p + 1$ is a prime power. Therefore this conjecture is certainly not true in general and needs to be restricted.

Conjecture 5.4. In $\text{Vec}_{\mathbb{Z}_p}$, the algebra $\mathbf{1} \oplus \text{Vec}_{\mathbb{Z}_p}$ admits a categorical Hopf algebra structure whose representation category is the near group category given by $G = \mathbb{Z}_p$ and multiplicity $m = |G| - 1$, if and only if $p = q^m - 1$ for some prime q .

Mathematica computations show that there is no Hopf algebra structure on the algebra $\mathbf{1} \oplus \text{Vec}_{\mathbb{Z}_5}$. Therefore, one cannot hope for a Hopf algebra on $\mathbf{1} \oplus \text{Vec}_{\mathbb{Z}_p}$ for all p either.

5.2 Condensation examples

We examine the condensation of some anyon theories which should give a large class of examples for extension by Hopf comonads. We will follow the notations defined in Subsection 3.2.4.

5.2.1 $\text{Rep}(S_3)$

A not so complicated condensation is that of $\mathcal{B} = \text{Rep}(S_3) = \{A, B, C\}$ with the condensable algebra $\mathcal{A} = A + C$. Notice that the fusion rules are $B \otimes B = A$, $C \otimes C = A + B + C$, $B \otimes C = C$ and A is the unit. Furthermore, we see that $\mathcal{A} = A + C$ is a condensable algebra as demonstrated in Appendix A.3. In this and the next examples, we follow the notations in [11, Table 1]. The condensation yields the category $\mathcal{B}_{\mathcal{A}} = \text{Vec}_{\mathbb{Z}_2}$ where the deconfined part is itself. To prove this, we use the Frobenius reciprocity. While this example is done explicitly, a very similar process is used for all other examples. Frobenius reciprocity for pairs $(X, Y) = \{(A, A), (A, B), (B, B), (A, C), (B, C), (C, C)\}$ implies, respectively:

- $D_{\mathcal{A}}(A)$ is a simple object as $\text{Hom}_{\mathcal{B}_{\mathcal{A}}}(A, A) = 1$ and we call it $\mathbf{1}$,
- $D_{\mathcal{A}}(B)$ contains no copy of $\mathbf{1}$,
- $D_{\mathcal{A}}(B)$ is actually a simple, we call it e ,
- $D_{\mathcal{A}}(C)$ has a copy of $\mathbf{1}$,
- $D_{\mathcal{A}}(C)$ has a copy of e ,
- $D_{\mathcal{A}}(C)$ has only two simples as $\text{Hom}_{\mathcal{B}_{\mathcal{A}}}(C, C) = 2$, therefore it is $\mathbf{1} + e$.

The fusion rules of $\mathbf{1}, e$ can be easily discovered by using the Frobenius reciprocity for pairs like $(B \otimes B, A), (B \otimes B, B)$, etc. The result is $\mathcal{B}_{\mathcal{A}} = \text{Vec}_{\mathbb{Z}_2}$.

Hence, the tensor functor is $D_{\mathcal{A}}(A) = \mathbf{1}, D_{\mathcal{A}}(B) = e, D_{\mathcal{A}}(C) = \mathbf{1} + e$, and the forgetful functor can be computed from the condition of right adjoint and it is given by $E_{\mathcal{A}}(\mathbf{1}) = A + C, E_{\mathcal{A}}(e) = B + C$. From these, it can be deduced $T_{\mathcal{A}}(X) = D_{\mathcal{A}}E_{\mathcal{A}}(X) = (2 + e) \otimes X$. As discussed in Subsection 3.2.4, $T_{\mathcal{A}}$,

being compositions of adjunctions, is automatically a Hopf comonad by general nonsense and it inherits the self-dual HM structure coming from Hopf algebra structure (which is unique) on $2 + e$ derived in Subsection 5.1.2.

5.2.2 $D(S_3)$

Consider the case $\mathcal{B} = D(S_3)$ and the condensable algebra $\mathcal{A} = A + C$. The objects are denoted by $\{A, B, C, D, E, F, G, H\}$, where $\{A, B, C\}$ is the canonical image of $\text{Rep}(S_3)$ in $D(S_3)$. By using the framework in Subsection 3.2.4, one derives the condensed category $\mathcal{B}_{\mathcal{A}} = \text{Rep}(D(\mathbb{Z}_2)) \oplus \{X, Y\}$ with the following fusion rules for X, Y ³⁾:

$$\begin{aligned} mX = Y, \quad mY = X, \quad \psi Y = X, \quad \psi X = Y, \quad eY = Y, \\ X^2 = \mathbf{1} + e + Y, \quad XY = m + \psi + X, \quad Y^2 = \mathbf{1} + e + Y, \end{aligned} \tag{5.9}$$

where $\text{Rep}(D(\mathbb{Z}_2)) = \{\mathbf{1}, e, m, \psi\}$ is the doubled \mathbb{Z}_2 theory, $\{\mathbf{1}, e\}$ is the image of the canonical embedding of $\text{Rep}(\mathbb{Z}_2) = \text{Vec}_{\mathbb{Z}_2}$, and exchanging m, e is a symmetry of the category. The maps $E_{\mathcal{A}}$ and $D_{\mathcal{A}}$ are given by

$$E_{\mathcal{A}} : \mathbf{1} \rightarrow A + C, \quad e \rightarrow B + C, \quad m \rightarrow D, \quad \psi \rightarrow E, \quad X \rightarrow D + E, \quad Y \rightarrow F + G + H, \tag{5.10}$$

$$D_{\mathcal{A}} : A \rightarrow \mathbf{1}, \quad B \rightarrow e, \quad C \rightarrow \mathbf{1} + e, \quad D \rightarrow m + X, \quad E \rightarrow \psi + X, \quad F, G, H \rightarrow Y. \tag{5.11}$$

Therefore, a straightforward calculation yields the Hopf comonad $T_{\mathcal{A}} = D_{\mathcal{A}}E_{\mathcal{A}}$ to be acting by $(2+e) \otimes -$ on the objects $\mathbf{1}, e, Y$ and $(\mathbf{1} + Y) \otimes -$ on m, ψ, X . It is easy to see that $\{X, Y\}$ can be replaced by $\{m + \psi, \mathbf{1} + e\}$ as they are modules of $\mathbf{1} + e$ and together with $\text{Rep}(D(\mathbb{Z}_2)) = D(\text{Vec}_{\mathbb{Z}_2})$ form the modules of $\mathbf{1} \oplus (\mathbf{1} + e)$ acting on $D(\text{Vec}_{\mathbb{Z}_2})$ as an HA with the structure in Subsection 5.1.2.

5.2.3 Ising \boxtimes Ising' from the gauging of $\text{SO}(8)_1$ followed by a condensation

There is an S_3 symmetry on $\text{SO}(8)_1$ which can be gauged (as in [12, Subsection 6.2]) giving the braided category $(\text{SO}(8)_1)_{S_3}^{\times, S_3}$ which has objects $A, B, C, X, \alpha X, \alpha^* X, Y_+, Y_-, X^{++}, X^{+-}, X^{--}, X^{-+}$ with fusion rules described in Appendix A.4. One can condense $A + C$, where A, B and C are the same as A, B and C in $\text{Rep}(S_3)$. The fusion rules can be computed and it will give Ising \boxtimes Ising' and three defects, where Ising and Ising' are two of the eight Ising modular tensor categories with central charge $\frac{1}{2}$ and $\frac{7}{2}$, respectively.

5.2.4 Haag

This example is about the famous Haagerup fusion category, denoted as H_2 in [20], a UFC of rank 6 with three indecomposable module categories coming from Q -systems $\mathbf{1}, \mathbf{1} + \rho, \mathbf{1} + \alpha + \alpha^*$. For the Q -systems definition, see, e.g., [4, Subsection 3.2]. The corresponding bimodule fusion categories are denoted by H_2, H_1, H_3 in [20]. It is hoped that there is a category \mathcal{C} which would make the sequence $\mathcal{C} \rightarrow H_3 \rightarrow H_2$ a gauging process where the second functor is the left adjoint of the forgetful functor.

5.2.5 $D(\text{Fib})$

This example is conjectured to be part of a sequence of examples which should build the sequence $D(\text{Haag}_k)$ defined later in Subsection 6.3.3. For $k = 0$ one recovers $D(\text{Fib})$. The condensable algebra $\mathcal{A} = \mathbf{1}\mathbf{1} + \tau\bar{\tau}$ gives the condensed category $\mathcal{B}_{\mathcal{A}} = \text{Fib}$, where $\mathbf{1}$ is deconfined and τ is a defect. The action $T_{\mathcal{A}} = D_{\mathcal{A}}E_{\mathcal{A}}$ can be computed from

$$E_{\mathcal{A}} : \mathbf{1} \rightarrow \mathbf{1}\mathbf{1} + \tau\bar{\tau}, \quad \tau \rightarrow \mathbf{1}\bar{\tau} + \bar{\tau}\mathbf{1} + \tau\bar{\tau}, \tag{5.12}$$

$$D_{\mathcal{A}} : \mathbf{1}\mathbf{1} \rightarrow \mathbf{1}, \quad \tau\bar{\tau} \rightarrow \mathbf{1} + \tau, \quad \mathbf{1}\bar{\tau}, \bar{\tau}\mathbf{1} \rightarrow \tau \tag{5.13}$$

and is given by $(2 + \tau) \otimes -$ which has the HA structure described in Appendix A.2.

³⁾ The third author would like to thank Colleen Delaney for confirming the fusion rules of X, Y .

6 Generalized symmetries

In this section, we first define category symmetries and then outline a program to generalize the theory of extension and gauging to category symmetries. We show that a category symmetry is essentially a tensor functor.

We will work with module categories over a braided fusion category \mathcal{B} . But the theory generalizes to fusion category simply by replacing module categories with bimodule categories. In practice, we are really working with bimodule categories of \mathcal{B} , but only those coming from module categories using the braidings. In the example of $\text{Vec}_{\mathbb{Z}_2}$, two out of the six bimodule categories are such bimodules.

6.1 Category symmetry

Let \mathcal{F} be a fusion category with a complete representative set of simple objects $\{X_i\}$, and \mathcal{B} be a unitary braided fusion category. The monoidal category of module categories $\text{Modc}(\mathcal{B})$ contains the Picard group $\text{Pic}(\mathcal{B})$, which is isomorphic to the group of braided tensor auto-equivalences $\text{Aut}_{\otimes}^{\text{br}}(\mathcal{B})$ of \mathcal{B} . $\text{Modc}(\mathcal{B})$ should be regarded as the group algebra of $\text{Pic}(\mathcal{B})$, and therefore the subcategory $\text{MPic}(\mathcal{B}) \subset \text{Endo}(\mathcal{B})$ generated by all functors from $\text{Modc}(\mathcal{B})$ should be regarded as the group algebra of $\text{Aut}_{\otimes}^{\text{br}}(\mathcal{B})$.

The isomorphism between $\text{Aut}_{\otimes}^{\text{br}}(\mathcal{B})$ and $\text{Pic}(\mathcal{B})$ underlies the physical relation between symmetries and defects. A generalization of this isomorphism to HM symmetries will be a characterization of the endo-functors $\text{MPic}(\mathcal{B})$ that correspond to $\text{Modc}(\mathcal{B})$. The existence of such a correspondence is obvious from the physical interpretation of unitary modular categories as anyon models. Suppose \mathcal{M} is an indecomposable module category over a unitary modular category \mathcal{B} . Then \mathcal{M} forms a gapped boundary between \mathcal{B} and itself. Forming the double $\mathcal{B} \boxtimes \mathcal{B}^{\text{op}}$ of \mathcal{B} , we can drag an anyon in the top layer \mathcal{B} to the boundary, and then lift it to the bottom layer resulting a map $\theta_{\mathcal{M}} : \mathcal{B} \rightarrow \mathcal{B}$. The interesting mathematical question is to characterize all such resulting endo-functors $\text{MPic}(\mathcal{B})$ of \mathcal{B} .

The theory of extension and gauging for group symmetries is a lifting problem. Starting with a group homomorphism $\rho : G \rightarrow \text{Aut}_{\otimes}^{\text{br}}(\mathcal{B})$, we look for a lifting first to $\underline{\rho} : \underline{G} \rightarrow \underline{\text{Pic}}(\mathcal{B})$, and then $\underline{\underline{\rho}} : \underline{\underline{G}} \rightarrow \underline{\underline{\text{Pic}}}(\mathcal{B})$. The generalization will be simply to replace $\text{Pic}(\mathcal{B})$ with $\text{MPic}(\mathcal{B})$.

Definition 6.1. 1. A category symmetry of a fusion category \mathcal{C} is a pair (\mathcal{F}, ρ) , where \mathcal{F} is a fusion category and $\rho : \mathcal{F} \rightarrow \mathcal{C}$ a tensor functor.

2. A Hopf symmetry of a fusion category \mathcal{C} is a Hopf monad $T : \mathcal{C} \rightarrow \mathcal{C}$.

Proposition 6.2. A Hopf symmetry T of a fusion category \mathcal{C} gives a category symmetry (\mathcal{C}^T, U) of \mathcal{C} , where $U : \mathcal{C}^T \rightarrow \mathcal{C}$ is the forgetful functor. Conversely, any category symmetry (\mathcal{F}, ρ) arises in such a way.

Proof. It is obvious from definitions in Subsection 3.1 and the fact that tensor functors on fusion categories have adjoints (see, e.g., [8, Subsection 1.3]). \square

6.2 Extension

In this part, we provide evidence for the *existence* of an extension theory based on Hopf symmetry, a generalization of the one based on group symmetry. Condensation as in Subsection 5.2 would also be contained as examples of this theory. This part is conjectural in nature and does not follow the theorem-proof format.

First we see how the group symmetry case can be recovered in the language of HMs. Most importantly for $T_G = \oplus T_g : \mathcal{C} \rightarrow \mathcal{C}$, why does the extension have a G -grading? The G -grading is the same as a Vec_G grading. Therefore, it is important to understand how to *recover* Vec_G from T_G .

The obvious solution seems to look at the modules of T_G , but this gives the equivariantization \mathcal{C}^G which, in spite of having $\text{Rep}(G)$ inside, does not contain Vec_G canonically. Most importantly, we would like something that has itself a grading “equal” to Vec_G .

The answer turns out to be the $T_G(\mathbf{1})$ -comodules. Indeed, just like any HM, $T_G(\mathbf{1})$ happens to be a coalgebra with the coalgebra structure $(T(\mathbf{1}), T_2(\mathbf{1}, \mathbf{1}), T_0)$ [9]. As $T_G(\mathbf{1}) = \oplus T_g(\mathbf{1}) = \oplus \mathbf{1}_g$ and the

coproduct $T_2(\mathbf{1}, \mathbf{1}): T_G(\mathbf{1}) \rightarrow T_G(\mathbf{1}) \otimes T_G(\mathbf{1})$ can be decomposed to $\mathbf{1}_g \rightarrow \mathbf{1}_g \otimes \mathbf{1}_g$, we have $\mathcal{C}^{\text{co-}T_G(\mathbf{1})} = \bigoplus_g \mathcal{C}^{\text{co-}T_g(\mathbf{1})}$.

Therefore, for a general HM T , a good starting point would be to consider the decomposition of $T(\mathbf{1})$ to simple coalgebras $T(\mathbf{1}) = \bigoplus m_i$ for the grading of the extension $\mathcal{C}_T^\times = \bigoplus \mathcal{C}_{m_i}$. For a Hopf comonad, one has to consider the decomposition of $T(\mathbf{1})$ to simple algebras as $T(\mathbf{1})$ is an algebra instead of a coalgebra.

Now we examine what each component \mathcal{C}_{m_i} should be. For the case of T_G , the m_i 's correspond to elements of G , and $\mathcal{C}_{m_i} = \mathcal{C}_g$ is known to be formed by objects of \mathcal{C} fixed by T_g . Since $m_i = T_g(\mathbf{1}) \cong \mathbf{1}$, each m_i -comodule is simply a copy of \mathcal{C} . Hence, we need to impose more restrictions on the objects that are allowed in the extension. A plausible generalization would be to somehow seek for a decomposition of the HM T into a sum of functors T_{m_i} , and require \mathcal{C}_{m_i} to consist of objects X of \mathcal{C} that are not only m_i -comodules but also satisfy $T_{m_i}(X) \cong m_i \otimes X$.

Let us see how this shapes the story for group and Hopf algebra symmetry. Notice that for groups, $m_i = \mathbf{1}_g$ and by construction we have the decomposition $T = \bigoplus T_g$. Thus the condition of being an m_i -comodule is trivial, while the nontrivial condition is that of $T_g(X) \cong \mathbf{1}_g \otimes X = X$, which exactly means X is fixed by T_g .

This is in contrast to $T(-) = \mathcal{H} \otimes -$ for a Hopf algebra \mathcal{H} where $T(\mathbf{1}) = \mathcal{H} = \bigoplus m_i$ decomposes to simple coalgebras m_i . One can obviously guess what T_{m_i} 's should be; they are defined as $T_{m_i}(-) = m_i \otimes -$ and $T = \bigoplus T_{m_i}$. Therefore the condition $T_{m_i}(X) \cong m_i \otimes X$ is trivial, while that being an m_i -comodule is not. Hence for the HM $T(-) = \mathcal{H} \otimes -$, the extension \mathcal{C}_T^\times should be essentially given by \mathcal{H} comodules.

Notice that for the Hopf comonads such as the ones found in the condensation examples, one needs to look at decompositions of $T(\mathbf{1})$ into simple algebras. As the HM $T = \mathcal{H} \otimes -$ is a Hopf comonad as well, one can as well consider algebra decomposition of \mathcal{H} as the grading for another extension. More generally, for self-dual HMs there are two ways of considering the extension.

For example, the Hopf comonad $T = (2 + e) \otimes -$, as demonstrated in Subsection 5.1.2, has the decomposition $\mathbf{1} \oplus (\mathbf{1} + e)$, and as mentioned in Subsection 5.2.2, this gives the extension given by $\mathcal{H} = 2 + e$ modules:

$$\mathcal{C}_T^\times = \mathcal{C}_\mathbf{1} \oplus \mathcal{C}_{\mathbf{1}+e} = D(\mathbb{Z}_2) \oplus \{X, Y\}.$$

How do we generalize the story to all HMs T ? First, one notes that $(T(X), T_2(\mathbf{1}, X))$ is always a $T(\mathbf{1})$ -comodule [9]. Therefore, it must decompose to m_i comodules. One can then define $T(X) = \bigoplus T_{m_i}(X)$ as being the decomposition to m_i comodules. Therefore, there is a definition for T_{m_i} : a certain functor from \mathcal{C} to $\mathcal{C}^{\text{co-}m_i}$ given by the composition of the functor $U : \mathcal{C} \rightarrow \mathcal{C}^{\text{co-}T(\mathbf{1})}$ where $U(X) = (T(X), T_2(\mathbf{1}, X))$, followed by the projection onto $\mathcal{C}^{\text{co-}m_i}$.

The next step is a crucial observation derived from the identities of the fusion operator $H_{\mathbf{1}, X}^l : T^2(X) \rightarrow T(\mathbf{1}) \otimes T(X)$. This is an isomorphism by the definition of an HM, and further the fusion operator is known to satisfy [7, Proposition 2.6] $(T_2(X, Y) \otimes \text{id}_{T(Z)})H_{X \otimes Y, Z}^l = (\text{id}_{T(X)} \otimes H_{Y, Z}^l)T_2(X, Y \otimes T(Z))$. Replacing X, Y by $\mathbf{1}$ and Z by X , implies that $H_{\mathbf{1}, X}^l$ is a $T(\mathbf{1})$ -comodule map between $(T^2(X), T_2(\mathbf{1}, T(X)))$ and $(T(\mathbf{1}) \otimes T(X), T_2(\mathbf{1}, \mathbf{1}) \otimes \text{id}_{T(X)})$. Therefore, as $T(\mathbf{1})$ decomposes, we obtain an isomorphism between their decompositions into m_i -comodules given by $T_{m_i}(T(X)) \xrightarrow{\cong} m_i \otimes T(X)$. The above is *always* true, and it shows that the idea of fixed point generalization as $T_{m_i}(Y) \cong m_i \otimes Y$ makes sense, as all $Y = T(X)$ satisfy this property. This can be also checked for $T = T_G$ where indeed T_g fixes any $T_G(X)$.

Notice the above discussion was for all *left* $T(\mathbf{1})$ -comodules, and the same can be said about right $T(\mathbf{1})$ -comodules by using the right fusion operator.

Treating $\mathbf{1}$ as a coalgebra with the apparent coalgebra structure, the map $\eta_\mathbf{1} : \mathbf{1} \rightarrow T(\mathbf{1})$ becomes an injective coalgebra morphism by the compatibility conditions between T_2 , T_0 , and η [9] (replacing X, Y with $\mathbf{1}$):

$$T_2(X, Y)\eta_{X \otimes Y} = \eta_X \otimes \eta_Y \quad \text{and} \quad T_0\eta_\mathbf{1} = 1. \tag{6.1}$$

This means $\mathbf{1}$ is a simple subcoalgebra inside $T(\mathbf{1})$, implying that in the decomposition of $T(\mathbf{1})$ into simple coalgebras one of the m_i 's can be chosen to be $\mathbf{1}$. We *identify* $\mathbf{1}$ with its image in $T(\mathbf{1})$ under $\eta_\mathbf{1}$.

It is conjectured (see the next paragraph) that the associated functor T_1 is the identity functor, in which case we have $\mathcal{C}_1 = \mathcal{C}$ and therefore \mathcal{C}_T^\times is an *extension* of \mathcal{C} .

We claim that $T_1(X)$ contains X . First of all, any X is trivially a comodule of $\mathbf{1}$. The embedding $\eta_1 : \mathbf{1} \rightarrow T(\mathbf{1})$ induces a $T(\mathbf{1})$ -comodule structure on X , where the comodule map is given by $\eta_1 \otimes \text{id}_X$. Then it is direct to see that the map $\eta_X : X \rightarrow T(X)$ is a $T(\mathbf{1})$ -comodule morphism by the identity $T_2(\mathbf{1}, X)\eta_X = \eta_1 \otimes \eta_X$ which is obtained from the first identity in (6.1) with X, Y replaced by $\mathbf{1}, X$, respectively. η_X is clearly also injective. Hence, X is a sub comodule of $T(X)$. Moreover, by the same identity as above, $T_2(\mathbf{1}, X)$ sends the image of X to

$$\mathbf{1} \otimes T(X) \subset T(\mathbf{1}) \otimes T(X).$$

But $\mathbf{1} \otimes T(X)$ is also where $T_1(X)$ is exclusively sent by $T_2(\mathbf{1}, X)$ due to the comodule decomposition. Hence $X \subset T_1(X)$ and it is conjectured to be equal.

One important aspect in the extension theory concerns the fusion rules. For the group case, the fusion rules correspond to possible liftings of an action $G \rightarrow \text{Aut}_\otimes(\mathcal{C})$ to $\underline{G} \rightarrow \underline{\text{Aut}}_\otimes(\mathcal{C})$. Notice that a lifting is presumed from the *beginning* as one states that T_G is an HM coming from $T_- : \underline{G} \rightarrow \underline{\text{Aut}}_\otimes(\mathcal{C})$. Therefore it will not be surprising to see that there is always a fusion rule for \mathcal{C}_T^\times although there might be more than one possible fusion rule just like in the group case. The Hopf algebra case should be similar, where the extension is given by the T -comodule category which a fusion category as T is also a Hopf comonad.

As for the general case, the fusion $\mathcal{C}_{m_i} \otimes \mathcal{C}_{m_j}$ is likely to be given by the multiplication $\mu : T^2 \rightarrow T$ whose restriction to $T_{m_i}(T_{m_j}(X))$ should give an indication of how \mathcal{C}_{m_i} and \mathcal{C}_{m_j} must fuse. Putting it more abstractly, if $T(\mathbf{1})$ happens to be a bialgebra and hence has a monoidal comodule category, one seeks a monoidal functor from the category of $T(\mathbf{1})$ -comodules to the monoidal category $\text{Bimod}(\mathcal{C})$, as all \mathcal{C}_{m_i} are \mathcal{C} -bimodules by fusion from left and right.

As for the associator, in the group case it is given by some vanishing obstruction in the fourth cohomology and classified by the third cohomology $H^3(G, U(1))$, and there is likely a similar story for the general case. It is known that deformations of the associator are given by the third Yetter-Davydov cohomology (see, e.g., [15, Chapter 7.22]).

Applying this reasoning to an extension of Vec given by an ordinary Hopf algebra \mathcal{H} , we should be looking at $H^i(\mathcal{H}\text{-comod})$ which for cocommutative hopf algebras turn out to be $HH^i(\mathcal{H}, \mathbb{C})$, the Hochschild cohomology of \mathcal{H} . As an example, notice that the trivial group symmetry in the case of Vec coincides with that of the action of the cocommutative Hopf algebra $\mathbb{C}[G]$. Hence, taking $\mathcal{H} = \mathbb{C}[G]$, this results in the cohomology $HH^i(\mathbb{C}[G], \mathbb{C})$ which is known to be the cohomology of G (see, e.g., [15, Chapter 7.22]), making the story compatible with the group extension.

A final piece of the puzzle is the extension of T to an HM T^\times on \mathcal{C}^\times . For the group case, it is known to be unique [16], while it might not be so for the Hopf algebra case.

Indeed, as mentioned before, for a Hopf algebra action $T(-) = \mathcal{H} \otimes -$ as an HM (Hopf comonad), the extension \mathcal{C}_T^\times is nothing but the T -comodule(-module) category itself. Generally for a self dual HM T , a canonical extension of T to its module category \mathcal{C}^T is always possible by $T^\times : (M, r) \rightarrow (T(M), T(r))$. This can be easily shown to be a self-dual HM and also obviously an *extension* of T acting on the copy of \mathcal{C} inside \mathcal{C}^T given by modules (X, ϵ_X) with ϵ the counit in Definition 3.3.

Now consider the condensation process in Subsection 5.2.2, where we switch to Hopf comonad $T_{\mathcal{A}}$ acting on $\mathcal{B}_{\mathcal{A}}$. From its appearance, $T_{\mathcal{A}}$ does not seem to be an extension of $T = (2 + e) \otimes (-)$ until one considers the replacement of X, Y by $m + \psi, \mathbf{1} + e$. But $T_{\mathcal{A}}$ is certainly not the canonical extension given above. Indeed for the canonical extension T^\times acting on module $(M, r) = (m, \epsilon_m)$, one obtains

$$T^\times((M, r)) = (2m + \psi, \epsilon_{2m+\psi}),$$

which is not the result of $T_{\mathcal{A}}$ acting on m by $\mathbf{1} + Y$ even at the object level. In fact, it is not hard to prove that $T^\times = (2 + e) \otimes$. This shows the possibility of different extensions of T itself.

There are many questions for which there are hints for the answers and are yet to be explored. We will explore a general HM extension theory in future work.

6.3 Gauging applications

6.3.1 A generalization of Fib to all primes Fib_p

Regarding Fib as a fusion category, we consider its generalization to a sequence of near-group categories [29] whose fusion rules would be denoted by Fib_p for all primes p , with $p = 1$ understood as Fib.

There are $p + 1$ isomorphic classes of simple objects denoted by \mathbb{Z}_p and X with the non-group fusion rule: $g \otimes X = X \otimes g = X$, $X \otimes X = (\bigoplus_{g \in \mathbb{Z}_p} g) \oplus pX$.

For $p = 2$, this is the well-studied fusion category $\frac{1}{2}E_6$, and for $p = 3$, this is the Izumi-Xu fusion rule; see also [18, p. 589] for more examples and a classification of near-group categories.

Open Question 6.3. Is there an HM on $\text{Vec}_{\mathbb{Z}_p}$ whose extension realizes Fib_p for each prime p ?

6.3.2 A generalization of the Haagerup category Haag_p

Fib as a fusion category fits into another potential sequence of fusion categories whose fusion rules will be denoted by Haag_p . Haag_p has $2p$ classes of simple objects denoted by $\alpha^i, i = 0, 1, \dots, p - 1$, and $\rho_i, i = 0, 1, \dots, p - 1$, where $\alpha^0 = \mathbf{1}, \rho_0 = \rho$. The α^i 's obey \mathbb{Z}_p fusion rule. The non-group fusion rules are determined by $\alpha^i \otimes \rho = \rho_i = \rho \otimes \alpha^{p-i}, \rho^2 = \mathbf{1} \oplus \sum_{i=0}^{p-1} \rho_i$.

For $p = 2$, this is the fusion rule of $\text{PSU}(2)_6$, and for $p = 3$, this is the Haagerup fusion rule.

Open Question 6.4. Is there an HM on $\text{Vec}_{\mathbb{Z}_p}$ with an extension that realizes Haag_p for each prime p ?

6.3.3 The doubled Haagerup category $D(\text{Haag}_p)$

The hypothetical modular category $D(\text{Haag}_p)$, defined in [17], for all odd prime p is of rank $p^2 + 3$ with anyons $\mathbf{1}, b, a_h, d_l$ with $1 \leq h \leq \frac{p^2-1}{2}, 1 \leq l \leq \frac{p^2+3}{2}$ of quantum dimensions $1, p\delta + 1, p\delta + 2, p\delta$, and $\delta = \frac{p+\sqrt{p^2+4}}{2}$ satisfying $\delta^2 = 1 + p\delta$. It is known to exist for $p \leq 13$ [18].

Proposition 6.5. *The object $\mathcal{A} = \mathbf{1} + b$ of $D\text{Haag}$ has a condensable algebra structure.*

Proof. $D\text{Haag}$ is weakly monoidal Morita equivalent to $\text{Haag} \boxtimes \text{Haag}^{\text{op}}$ [25], and $\mathbf{1} + \rho \boxtimes \rho^*$ is a condensable algebra, which is sent to $\mathbf{1} + b$. □

Consider the condensable algebra $\mathcal{A} = \mathbf{1} + b$, and then we have⁴⁾

$$D_{\mathcal{A}}(\mathbf{1}) = \mathbf{1}, \quad D_{\mathcal{A}}(b) = \mathbf{1} + X, \quad D_{\mathcal{A}}(a_h) = X + \alpha_{i,j} + \alpha_{i,j}^*, \quad D_{\mathcal{A}}(d_l) = X,$$

where $\alpha_{i,j}$ form $D(\mathbb{Z}_p)$, and $X^2 = D(\mathbb{Z}_p) + p^2X$. So $\mathcal{C} = D(\mathbb{Z}_p) \oplus \{X\}$ with deconfined $\mathcal{D} = D(\mathbb{Z}_p)$. It follows that the HM is

$$T_{\mathcal{A}}(\mathbf{1}) = 2 + X, \quad T_{\mathcal{A}}(\alpha_{ij}) = \alpha_{i,j} + \alpha_{i,j}^* + X, \quad T_{\mathcal{A}}(X) = D(\mathbb{Z}_p) + (p^2 + 2)X.$$

So $T_{\mathcal{A}}(a) = a + a^* + X \otimes a$ is an HM on $D(\mathbb{Z}_p) \oplus \{X\}$, and the question in general is what T is on $D(\mathbb{Z}_p)$?

Open Question 6.6. Can $D(\text{Haag}_p)$ be realized through gauging a Hopf monad symmetry on $D(\mathbb{Z}_p)$?

6.4 The tensor product of modules of $D(\mathbb{Z}_N)$

An important topic to study for Hopf monad extension is that of the monoidal category $\mathcal{C}_T^\times = \bigoplus \mathcal{C}_{m_i}$ formed by the bimodules \mathcal{C}_{m_i} . As a large class of examples involve the double of abelian groups \mathbb{Z}_N , we have decided to include a study of the product of the modules of $D(\mathbb{Z}_N)$ when viewed as a bimodule using the braiding structure. It is important to note that most of the work is already done in [16, Proposition 3.19], where one characterizes the bimodules of Vec_A for abelian group A and computes their tensor product over Vec_A . Indeed, in this case $A = \mathbb{Z}_N \times \mathbb{Z}_N$, and we only need to understand which bimodule corresponds to the module given a bimodule structure by braiding. Although in this section, [16, Proposition 3.19] is used directly without any explanation, in Appendix A.1, we form a table of $\text{Vec}_{\mathbb{Z}_2}$ -bimodule monoidal category by going explicitly through the construction of the fusion of

⁴⁾ The third author thanks Terry Gannon for computing the fusion rules of the condensed category of $D\text{Haag}$.

bimodules and give more details of the calculations in this section. A problem close to this, is the tensor product of $\text{Vec}_{\mathbb{Z}_p}$ bimodules, which is analyzed in [3].

We refer to the discussion at the beginning of Section 5. Consider a module $\mathcal{M} = \mathcal{M}(A, \rho)$ of $\mathcal{C} = D(\mathbb{Z}_N)$ coming from the algebra $A \subset \mathbb{Z}_N \times \mathbb{Z}_N$ with a 2-cocycle $\rho \in H^2(A, \mathbb{C})$. Notice for abelian groups 2-cocycles and skew-symmetric bicharacters form the same group (see, e.g., [16, p. 12]). We will assume that ρ is trivial to simplify calculations but it is easy to do the general case as well. The right action of $\mathcal{C} = D\mathbb{Z}_N$ on \mathcal{M} is given by $M.X := X.M$, i.e., just the left action, which is actually fusion as \mathcal{M} is really inside \mathcal{C} . Recall the associativity constraint for the left action denoted by

$$m_{X,Y,M} := (X \otimes Y).M \rightarrow X.(Y.M).$$

The braiding comes in the associativity constraints for the right action, given by

$$n_{M,X,Y} := M.(X \otimes Y) \rightarrow (M.X).Y$$

and defined as $c_{X,Y}$ as the left-hand side is $(X \otimes Y).M$ and the right-hand side is $Y.(X.M)$. Of course, we are using the facts that the left action is given by fusion and that the associator α of \mathcal{C} which also gives the associator $m_{X,Y,M}$ of the left action is trivial. Otherwise, for the general case of braided \mathcal{C} , the right action is still defined as $M.X := X.M$, but with associativity constraint $n_{M,X,Y} = m_{Y,X,M}(c_{X,Y} \otimes \text{id}_M)$. The final structure map, the *middle* associativity constraint $b_{X,M,Z} : (X.M).Z \rightarrow X.(M.Z)$ is $m_{X,Z,M}c_{Z,X}m_{Z,X,M}^{-1}$ for the general case of a braided \mathcal{C} . The reason that these structure maps (n) b satisfy the commuting diagram constraints for the definition of (right module) bimodule [15, Definition 7.1.7] can be derived by using the similar diagrams for the definition of the braided category \mathcal{C} and the \mathcal{M} as a module of \mathcal{C} . Although in the case of $D\mathbb{Z}_N$, one can see that m is the associator α of \mathcal{C} , and all diagrams are formed by braiding and the associator which commute due to the axioms satisfied by \mathcal{C} itself.

By standard module category facts, \mathcal{M} as a bimodule of $D\mathbb{Z}_N$, is equivalent to a left module of $D\mathbb{Z}_N \boxtimes D\mathbb{Z}_N^{\text{op}}$. The associator for this action is given by

$$l_{X_1 \boxtimes Y_1, X_2 \boxtimes Y_2, M} := ((X_1 \otimes X_2) \boxtimes (Y_1 \otimes_{\text{op}} Y_2)).M \rightarrow (X_1 \boxtimes Y_1).((X_2 \boxtimes Y_2).M) \tag{6.2}$$

i.e.,

$$((X_1 \otimes X_2).M).(Y_2 \otimes Y_1) \rightarrow (X_1).((X_2.M).Y_2).Y_1, \tag{6.3}$$

$$l_{X_1 \boxtimes Y_1, X_2 \boxtimes Y_2, M} = b_{X_1, X_2, M, Y_2} n_{X_1, (X_2, M), Y_2, Y_1} m_{X_1, X_2, M}. \tag{6.4}$$

In our case this gives $c_{Y_2, X_1} c_{Y_2, Y_1}$. The module \mathcal{M} is given by the algebra A , and we need to see which algebra B is in $D(\mathbb{Z}_N) \boxtimes D(\mathbb{Z}_N) = \text{Vec}_{\mathbb{Z}_N^2 \oplus \mathbb{Z}_N^2}$ (as a fusion category, which is why the superscript “op” is omitted), and which 2-cocycle ψ on B would give us the module on $\text{Vec}_{\mathbb{Z}_N^2 \oplus \mathbb{Z}_N^2}$ corresponding to \mathcal{M} as a bimodule on $D(\mathbb{Z}_N)$. Notice that the fusion category structure is what determines the category of (bi)modules, and that is why one can ignore braiding and omit “op”. The algebra B is essentially the stabilizer of a simple object like $[A]$, the trivial coset, inside \mathcal{M} (see [16, Subsection 2.7]):

$$(X \otimes Y).[A] = (X.[A]).Y = Y.(X.[A]) = y + x + [A], \tag{6.5}$$

where the summation on the right is by looking at what elements X, Y correspond to in the abelian group. To have the coset fixed, it is obvious that one needs to have $x + y \in A$. Therefore, the algebra B is given by all those pairs $(x, y) \in \mathbb{Z}_N^2 \oplus \mathbb{Z}_N^2$ with $x + y \in A$. The 2-cocycle is derived from the associator by restricting to B (as in [16, Subsection 2.7]). Therefore $\psi((x_1, y_1), (x_2, y_2)) = c_{Y_2, X_1} c_{Y_2, Y_1} = c_{Y_2, X_1 \otimes Y_1}$. This is indeed a bicharacter on B as $c_{a,b} = e^{\frac{2\pi i}{N} a_2 b_1}$, where $a = (a_1, a_2) \in \mathbb{Z}_N^2$ and similarly for $b = (b_1, b_2)$. Hence,

$$\psi((x_1, y_1), (x_2, y_2)) = e^{\frac{2\pi i}{N} y_{22}(x_{11} + y_{11})}.$$

The next step is to take this 2-cocycle and make it into a *skew-symmetric* bicharacter, as only then one can use the theorem necessary to find the result of the bimodule product. Adding a suitable 2-coboundary $d\epsilon$ makes this change possible, where $\epsilon(g) = \psi(g, g)^{\frac{1}{2}}$ giving us

$$(d\epsilon)(g_1, g_2) = \psi(g_1, g_1)^{\frac{1}{2}} \psi(g_2, g_2)^{\frac{1}{2}} \psi(g_1 + g_2, g_1 + g_2)^{-\frac{1}{2}}.$$

Notice in general that any 2-cocycles ω (for abelian groups) are bicharacters which implies $\omega(2g, 2g) = \omega(g, g)^4$ and the above calculation can be done for any ω to make it a skew-symmetric bicharacter. The new skew-symmetric bicharacter is

$$\psi(g_1, g_2)d\epsilon(g_1, g_2) = \psi(g_1, g_2)\psi(g_1, g_1)^{\frac{1}{2}}\psi(g_2, g_2)^{\frac{1}{2}}\psi(g_1 + g_2, g_1 + g_2)^{-\frac{1}{2}} = \psi(g_1, g_2)^{\frac{1}{2}}\psi(g_2, g_1)^{-\frac{1}{2}}$$

which, by abuse of notation, $\psi((x_1, y_1), (x_2, y_2)) = e^{\frac{2\pi i}{N}(y_{22}(x_{11}+y_{11})-(x_{21}+y_{21})y_{12})}$. If the bicharacter ρ on A was not trivial, then one would have to include in the associator l the product $\rho(X_1, X_2)\rho(Y_1, Y_2)$ and the rest would be similar.

As an application, consider the module \mathcal{M} on $D(\mathbb{Z}_2)$ given by the algebra $A = 0 \times \mathbb{Z}_2$ and the trivial bicharacter. Then \mathcal{M} as a bimodule on $D(\mathbb{Z}_2)$ is a module on $\text{Vec}_{\mathbb{Z}_2^2 \oplus \mathbb{Z}_2^2}$ with the corresponding algebra $B = \{(x, y) \mid x + y \in 0 \times \mathbb{Z}_2\}$ which is

$$(0 \times \mathbb{Z}_2, 0 \times \mathbb{Z}_2) \cup (1 \times \mathbb{Z}_2, 1 \times \mathbb{Z}_2). \tag{6.6}$$

Furthermore, the bicharacter ψ is also trivial as $x + y \in 0 \times \mathbb{Z}_2$ implies $x_{11} + y_{11} = x_{21} + y_{21} = 0$. These together with [16, Proposition 3.19], imply $\mathcal{M} \boxtimes_{D(\mathbb{Z}_2)} \mathcal{M} = 2\mathcal{M}$. The fusion structure of bimodules is not necessarily *symmetric* with respect to the algebras. For example, for $A = \mathbb{Z}_2 \times 0$ giving a bimodule \mathcal{M} ,

$$\mathcal{M} \boxtimes_{D(\mathbb{Z}_2)} \mathcal{M} = 2\mathcal{M}((\mathbb{Z}_2 \times 0, \mathbb{Z}_2 \times 0), \psi_{\text{trivial}}),$$

where the product gives the bimodule corresponding to the algebra $(\mathbb{Z}_2 \times 0, \mathbb{Z}_2 \times 0) \subset \mathbb{Z}_2^2 \oplus \mathbb{Z}_2^2$ with the trivial bicharacter. Details are in Appendix A.1.

6.5 Applications to physics

The relation between symmetry and defect in the group symmetry case should be extended to the generalized symmetry and defect case. We will leave such a physics theory as [2, 12] to the future.

7 Modular categories as a tetra-category

Regarding gauging HMs as a new method to construct modular categories, we use gauging to organize modular categories for a structure theory with an eye towards a classification.

7.1 Weak Hopf monads

From the definition of category symmetry, a fusion category \mathcal{F} is a symmetry of \mathcal{C} , and then \mathcal{C} is also a module category of \mathcal{F} by using the tensor functor to induce the action: $Y.X := \rho(Y) \otimes X, Y \in \mathcal{F}, X \in \mathcal{C}$. Since Vec is not a Fib module category, it follows that Fib cannot be a symmetry of Vec . Therefore, we cannot regard $D(\text{Fib})$ as a gauging of a Fib symmetry on Vec .

There is a generalization of HMs to weak Hopf monads [5]. Then for a weak Hopf algebra \mathcal{H} , the functor $T_{\mathcal{H}}$ is a weak Hopf monad on Vec and its modules can realize *any* fusion theory. Hence, a weaker notion of category symmetry based on weak Hopf monads could make Fib to be a category symmetry of Vec as it is expected in Example 5.2.5. We will leave such a theory to the future and use such a possibility to speculate on a structure theory of modular categories.

Definition 7.1. 1. A weak category symmetry of a fusion category \mathcal{C} is a pair (\mathcal{F}, ρ) , where \mathcal{F} is a fusion category and $\rho : \mathcal{F} \rightarrow \mathcal{C}$ a strong Frobenius functor.

2. A weak Hopf symmetry of a fusion category \mathcal{C} is a weak Hopf monad $T : \mathcal{C} \rightarrow \mathcal{C}$.

7.2 Simple modular category

Definition 7.2. A modular category \mathcal{B} is simple if \mathcal{B} is prime (not a Deligne tensor product) and contains no non-trivial normal (or condensable) algebras.

7.2.1 Simple pointed categories

Classification of all simple MCs is certainly a hard problem. We could start to find all simple pointed MCs. These are coming from abelian groups G with some non-degenerate quadratic forms $q : G \rightarrow \mathbb{R}/\mathbb{Z}$ which are classified in [19, Theorem 5.4] where q satisfies $\theta_a = e^{2\pi i q(a)}, \forall a \in G$. One needs to first look for those q 's which do not have a kernel, and hence as $q(a) = c(a, a)$ (following the notations of [19]) we must have $c(a, a) \neq 0, \forall 0 \neq a \in G$. Those would be the (G, q) that cannot have a commutative algebra and therefore, a condensable algebra. This is easy following the classification result for the prime abelian anyonic theory. First we list all classes here as presented in [19, Theorem 5.4]:

1. If $p \neq 2$ and $\epsilon = \pm 1, \omega_{p,k}^\epsilon$ denotes the abelian anyon with fusion rules given by $\mathbb{Z}/p^k\mathbb{Z}$ and abelian 3-cocycle $(0, c)$, where $c(x, y) = \frac{uxy}{p^k}$, for some $u \in \mathbb{Z}^{>0}$ with $(p, u) = 1$ and $(\frac{2u}{p}) = \epsilon$, where $(\frac{\cdot}{p})$ is the Legendre symbol.

2. If $\epsilon \in (\mathbb{Z}/8\mathbb{Z})^\times, \omega_{2,k}^\epsilon$ denotes the abelian anyon with fusion rules given by $\mathbb{Z}/2^k\mathbb{Z}$ and abelian 3-cocycle

$$c(x, y) = \frac{uxy}{2^{k+1}}, \quad \omega(x, y, z) = \begin{cases} \frac{1}{2}, & \text{if } x = y = z = 2^{k-1}, \\ 0, & \text{otherwise} \end{cases}$$

for some $u \in \mathbb{Z}^{>0}$ with $u \equiv \epsilon \pmod 8$. The abelian anyons $w_{2,k}^1$ and $w_{2,k}^{-1}$ are defined for all $k \geq 1$ and $w_{2,k}^5$ and $w_{2,k}^{-5}$ for all $k \geq 2$.

3. E_k denoted the abelian anyon with fusion rules given by $\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}$ and abelian 3-cocycle $(0, c)$, where $c \in \text{Hom}(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ is defined by

$$c(\vec{e}_i, \vec{e}_j) = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{2^k}, & \text{if } i \neq j. \end{cases}$$

4. F_k denoted the abelian anyon with fusion rules given by $\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}$ and abelian 3-cocycle $(0, c)$, where $c \in \text{Hom}(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ is defined by

$$c(\vec{e}_i, \vec{e}_j) = \begin{cases} \frac{1}{2^{k-1}}, & \text{if } i = j, \\ \frac{1}{2^k}, & \text{if } i \neq j. \end{cases}$$

We note that for the first class $\sum_{j=0}^{p-1} jp^{k-1}$ can be easily checked to be a nontrivial condensable algebra if $k > 1$. For $k = 1$ there are no trivial twists except for the unit object and hence this gives simple modular categories. The second class is similar to the first in this regard except that k needs to be greater than two to have trivial twists and a condensable algebra $\sum_{j=0}^{p-1} jp^{k-1}$.

The third class has always the condensable algebra $\sum_{j \in E_k} j$ as all twists are trivial. This is in contrast with the fourth class where all F_k 's are simple modular categories without any trivial twist.

7.2.2 $SU(2)_k$

Another set of examples would be to find the simple modular categories among $\mathfrak{su}(2)_k$. The simples are labelled by the common notation $0, \dots, k$. Here, we know that the only possible normal algebra is $0 + k$ as $\theta_k = e^{\frac{2\pi i k}{4}}$ and that can only be 1 when $k \equiv 0 \pmod 4$. Otherwise for k odd, the anyon k is a semion and we have a full modular subcategory *semi* given by the unit and the semion. Hence, by [26], the category splits to two MCs, and the splitting is of the form *semi* \times $PSU(2)_k$. The case which remains to prove that does not split to a product and is therefore simple, is $k \equiv 2 \pmod 4$. This follows from the ADE classification of modular invariants of $SU(2)_k$ (see, e.g., [22, 27]), with the exception of $k = 10$, which has a boson for anyon with label 6 [27, p. 220] giving the condensible algebra $0 + 6$.

Open Question 7.3. 1. Is it true that simple modular categories are determined by modular data? Note that this is false for general modular categories [6, 14, 24].

2. Is it true the number of simple modular categories for a fixed rank has a polynomial growth as the rank goes to infinity? Note that this is false for all modular categories [10].

3. Can we define a zeta function for all modular categories using simple modular categories as analogues of primes?

7.3 The tetra-category of modular categories

We can imagine to use all modular categories to define a state sum invariant of 4-manifolds X . Fixing a generic cellulation X into 4-cells, we color each 4-cell by a modular category and the 3-face between two of them by the fusion category resulting from condensing an algebra or the T -crossed extension; 2-simplices to be colored by bi-modules, etc. What resulted is essentially a promotion of the tri-category of all fusion categories to a tetra-category, which does not yet have an algebraic definition. Therefore, we will not attempt to make the idea rigorous in any way.

Heuristically, the tetra-category has the following:

1. 0-morphism or object: modular categories.
2. 1-morphism: condensable algebras or fusion categories.
3. 2-morphism: bimodules between fusion categories.
4. 3-morphism: module functors between bimodules.
5. 4-morphism: natural transformations between module functors.

Then a structure theory of modular categories is the same as an explicit description of this tetra-category.

Acknowledgements The first author was supported by the Simons Foundation. The third author was supported by National Science Foundation of USA (Grants Nos. DMS-1411212 and FRG-1664351).

References

- 1 Aissaoui S, Makhlouf A. On classification of finite-dimensional superbialgebras and Hopf superalgebras. ArXiv:1301.0838, 2013
- 2 Barkeshli M, Bonderson P, Cheng M, et al. Symmetry, defects, and gauging of topological phases. ArXiv:1410.4540, 2014
- 3 Barter D, Bridgeman J C, Jones C. Domain walls in topological phases and the Brauer-Picard ring for $\text{Vec}(\mathbb{Z}/p\mathbb{Z})$. ArXiv:1806.01279, 2018
- 4 Bischoff M, Kawahigashi Y, Longo R, et al. Tensor Categories and Endomorphisms of Von Neumann Algebras: With Applications to Quantum Field Theory, Volume 3. New York: Springer, 2015
- 5 Böhm G, Lack S, Street R. Weak bimonads and weak Hopf monads. *J Algebra*, 2011, 328: 1–30
- 6 Bonderson P, Delaney C, Galindo C, et al. On invariants of modular categories beyond modular data. *J Pure Appl Algebra*, 2018, in press
- 7 Bruguières A, Lack S, Virelizier A. Hopf monads on monoidal categories. *Adv Math*, 2011, 227: 745–800
- 8 Bruguières A, Natale S. Exact sequences of tensor categories. *Int Math Res Not IMRN*, 2011, 24: 5644–5705
- 9 Bruguières A, Virelizier A. Hopf monads. *Adv Math*, 2007, 215: 679–733
- 10 Bruillard P, Ng S-H, Rowell E, et al. Rank-finiteness for modular categories. *J Amer Math Soc*, 2016, 29: 857–881
- 11 Cong I, Cheng M, Wang Z. Topological quantum computation with gapped boundaries. ArXiv:1609.02037, 2016
- 12 Cui S X, Galindo C, Plavnik J Y, et al. On gauging symmetry of modular categories. *Comm Math Phys*, 2016, 348: 1043–1064
- 13 Cui S X, Hong S-M, Wang Z. Universal quantum computation with weakly integral anyons. *Quantum Inf Process*, 2015, 14: 2687–2727
- 14 Delaney C, Tran A. A systematic search of knot and link invariants beyond modular data. ArXiv:1806.02843, 2018
- 15 Etingof P, Gelaki S, Nikshych D, et al. *Tensor Categories*, Volume 205. Providence: Amer Math Soc, 2016
- 16 Etingof P, Nikshych D, Ostrik V, et al. Fusion categories and homotopy theory. *Quantum Topol*, 2010, 1: 209–273
- 17 Evans D E, Gannon T. The exoticness and realisability of twisted Haagerup-Izumi modular data. *Comm Math Phys*, 2011, 307: 463
- 18 Evans D E, Gannon T. Near-group fusion categories and their doubles. *Adv Math*, 2014, 255: 586–640
- 19 Galindo C, Jaramillo N. Solutions of the hexagon equation for Abelian anyons. *Rev Colombiana Mat*, 2016, 50: 277–298
- 20 Grossman P, Snyder N. Quantum subgroups of the Haagerup fusion categories. *Comm Math Phys*, 2012, 311: 617–643

- 21 Izumi M. A cuntz algebra approach to the classification of near-group categories. In: Proceedings of the 2014 Maui and 2015 Qinhuangdao Conferences in Honour of Vaughan FR Jones' 60th Birthday. Centre for Mathematics and Its Applications, Mathematical Sciences Institute. Canberra: The Australian National University, 2017, 222–343
- 22 Kirillov Jr A, Ostrik V. On a q -analogue of the McKay correspondence and the ade classification of \mathfrak{sl}_2 conformal field theories. *Adv Math*, 2002, 171: 183–227
- 23 Larson H K. Pseudo-unitary non-self-dual fusion categories of rank 4. *J Algebra*, 2014, 415: 184–213
- 24 Mignard M, Schauenburg P. Modular categories are not determined by their modular data. *ArXiv:1708.02796*, 2017
- 25 Müger M. From subfactors to categories and topology II: The quantum double of tensor categories and subfactors. *J Pure Appl Algebra*, 2003, 180: 159–219
- 26 Müger M. On the structure of modular categories. *Proc Lond Math Soc* (3), 2003, 87: 291–308
- 27 Ostrik V. Module categories over the Drinfeld double of a finite group. *Int Math Res Not IMRN*, 2003, 2003: 1507–1520
- 28 Ostrik V. Module categories, weak Hopf algebras and modular invariants. *Transform Groups*, 2003, 8: 177–206
- 29 Siehler J. Near-group categories. *Algebr Geom Topol*, 2003, 3: 719–775

Appendix A

A.1 Module categories over $\text{Vec}_{\mathbb{Z}_2}$

We want to calculate the monoidal category given by the bimodules of $\text{Vec}_{\mathbb{Z}_2}$. First, let us review the background material of [16, Proposition 3.19, Subsections 3.6 and 2.7]. We note that an alternative approach is presented in [3].

For G a general finite group, the indecomposable semisimple module category of Vec_G is labelled by $\mathcal{M}(H, \psi)$ where H is a subgroup and $\psi \in H^2(H, k^\times)$. There is an identification by conjugation of elements of G . For abelian groups which have no nontrivial conjugation, there is no identification. Recall that the rank of the module category corresponding to the pair (H, ψ) is equal to the number of right cosets.

From now on, groups like E are assumed to be abelian and therefore, 2-cocycles are the same as skew-symmetric bicharacters, i.e., biadditive maps $b : E \times E \rightarrow k^\times$ where $b(x, x) = 1$. We shall assume $k = \mathbb{C}$.

For $N \subset E$ a subgroup and ψ a bicharacter on E , denote the orthogonal complement of N with respect to ψ by N^\perp to be the subgroup $\{n \in E \mid \psi(x, n) = 1, \forall x \in N\}$. Notice if ψ is nondegenerate then N^\perp can be identified with E/N and $|N| \cdot |N^\perp| = |E|$. Also, the mere existence of a nondegenerate bicharacter implies $|E|$ being a square. In our case of study, most of the time $\psi = 1$ (as H^2 is trivial) and therefore $N^\perp = E$.

We will need to sometimes consider how a bicharacter can be pushed forward via a map. Consider abelian groups A and B with group homomorphism $\phi : B \rightarrow A$, and skew symmetric bicharacter ξ on B . Take $K = \ker \phi$ and K^\perp the orthogonal complement with respect to ξ . Then it can be shown that ξ can be (in the most obvious way) pushed forward to define a skew-symmetric bicharacter on $H = \phi(K^\perp)$.

Tensor product of bimodule categories. Let A_1, A_2 and A_3 be finite abelian groups (for example \mathbb{Z}_2). We would like to take the product of a $\text{Vec}_{A_1} - \text{Vec}_{A_2}$ module with a $\text{Vec}_{A_2} - \text{Vec}_{A_3}$ module and compute what the $\text{Vec}_{A_1} - \text{Vec}_{A_3}$ bimodule is. First, notice that a $\text{Vec}_A - \text{Vec}_B$ bimodule is simply an $A \oplus B^{\text{opp}}$ -module but B is abelian so an $A \oplus B$ -module. As semisimple indecomposable modules of Vec_G have been classified, we have to compute $\mathcal{M}(H, \psi) \boxtimes_{\text{Vec}_{A_2}} \mathcal{M}(H', \psi')$ for $H \subset A_1 \oplus A_2, H' \subset A_2 \oplus A_3$ and two skew-symmetric bicharacters ψ, ψ' on H, H' .

First, define $H \circ H'$ to be the subgroup of elements $(a_1, -a_2, a_2, a_3)$ in $H \oplus H'$. Let $H \cap H' \subset A_2$ be their intersection inside A_2 , i.e., $(H \cap A_2) \cap (H' \cap A_2)$; for example if $A_i = \mathbb{Z}_2$ and $H = \Delta$ the diagonal then $H \cap H' = \{0\}$, the trivial group as $\Delta \cap A_2 = \{0\}$. Next, consider $H \cap H'$ as a subgroup of $H \circ H'$ by the anti-diagonal embedding $x \rightarrow (0, -x, x, 0)$. Notice the bicharacter $\psi \times \psi'$ on $H \oplus H'$ can be restricted to $H \circ H'$. Hence, we can take the orthogonal complement $(H \cap H')^\perp$ of $H \cap H'$. Finally, as $H \cap H'$ is living inside the kernel of the projection $\phi : A_1 \oplus A_2 \oplus A_2 \oplus A_3 \rightarrow A_1 \oplus A_3$, we can push forward its bicharacter and get the subgroup $H'' = \phi((H \cap H')^\perp) \subset A_1 \oplus A_3$, with the bicharacter $\psi'' = \phi_*(\psi \times \psi')$. We can finally state [16, Proposition 3.19].

Proposition A.1. $\mathcal{M}(H, \psi) \boxtimes_{\text{Vec}_{A_2}} \mathcal{M}(H', \psi') = m \cdot \mathcal{M}(H'', \psi'')$, where

$$m = \frac{|H \cap H'| \cdot |(H \cap H')^\perp| \cdot |A_2|}{|H| \cdot |H'|}.$$

Let us study the case of $\text{Vec}_{\mathbb{Z}_2}$ bimodules when $A = A_i = \text{Vec}_{\mathbb{Z}_2}$. There are six pairs (H, ψ) for subgroups of $A \oplus A$. They give six bimodules $M_{i,j}$ where i will denote the rank (which is $2|A|/|H| = 4/|H|$) and j will be present if there are more than one instance of rank i :

1. $M_2 = \mathcal{M}(\Delta, 1)$ for the diagonal subgroup,
2. $M_{1,1} = \mathcal{M}(A \oplus A, 1)$ with the trivial bicharacter,
3. $M_{1,2} = \mathcal{M}(A \oplus A, \xi)$ with the nontrivial bicharacter,
4. $M_4 = \mathcal{M}(1, 1)$ with the trivial subgroup,
5. $M_{2,1} = \mathcal{M}(A \oplus \{0\}, 1)$,
6. $M_{2,2} = \mathcal{M}(\{0\} \oplus A, 1)$.

ξ can be computed. Notice $2x = 0, \forall x \in A \oplus A$ and $\xi(x, x) = 1$ (skew-symmetry), and hence

$$\xi(x, y)\xi(x, 0) = \xi(x, y + 0) \Rightarrow \xi(x, 0) = 1, \quad \text{similarly} \quad \xi(0, x) = 1, \tag{A.1}$$

$$\xi(x, y)^2 = \xi(x, 2y) = \xi(x, 0) = 1 \Rightarrow \xi(x, y) = \pm 1, \tag{A.2}$$

$$\xi(x, y) = \xi(x, y)\xi(y, y) = \xi(x + y, y) \Rightarrow \xi(x, y) = \xi(x, x + y) = \xi(x + y, y), \tag{A.3}$$

$$\xi(x, y) = \xi(x + y, y) = \xi(x + y, x + 2y = x) = \xi(x + y, x) = \xi(y, x) \Rightarrow \xi \text{ is symmetric.} \tag{A.4}$$

There are 16 elements in $A \oplus A$, but from the above, one needs to only determine ξ for the three possible pairs from the three elements $a = (1, 1), b = (1, 0), c = (0, 1) \in A$. But

$$\xi(a, b) = \xi(a, a + b) = \xi(a, c) = \xi(a + c, c) = \xi(b, c) = t.$$

So all their values are equal. Therefore, as t can be only ± 1 , for ξ to be nontrivial, we choose $t = -1$. This can be checked to satisfy all relations of a skew symmetric bicharacter.

From here, finding the multiplication table of the bimodule category is only straightforward calculation using what was described before the proposition. It is especially easy to do as in \mathbb{Z}_2 . We always have $a = -a$ and the negative signs can be ignored. Furthermore, for the $M_{i,j}$ with trivial character which gives orthogonal complement $N \subset E$ always equal to the whole group E , (H'', ψ'') in the proposition turns out to be simply $(\phi(H \circ H'), 1)$, that is the subgroup (a_1, a_3) for which there exists a_2 such that $(a_1, a_2) \in H, (a_2, a_3) \in H'$. Notice the construction of $H \circ H'$ already shows the category is not symmetric. M_2 is the unit as it can be observed from Table 1. The category can also be seen to be not rigid therefore certainly not a fusion category. This could have been guessed from the proposition where the tensor of two bimodules is a single one and if it is a fusion category then every element is invertible, the category is pointed and therefore a group and there should be no multiplicities at all. This is very unlikely and probably not true in most cases. In brief, the category is a semisimple tensor category.

$D(\mathbb{Z}_2)$ bimodules. As promised in Subsection 6.4, we would like to detail the computations used to compute the tensor product of the select bimodules of $D(\mathbb{Z}_2)$.

Table 1 $A \boxtimes_{\text{Vec}_{\mathbb{Z}_2}} B$ where A is from the left column

$\boxtimes_{\text{Vec}_{\mathbb{Z}_2}}$	M_2	$M_{1,1}$	$M_{1,2}$	M_4	$M_{2,1}$	$M_{2,2}$
M_2	M_2	$M_{1,1}$	$M_{1,2}$	M_4	$M_{2,1}$	$M_{2,2}$
$M_{1,1}$	$M_{1,1}$	$2M_{1,1}$	$M_{2,1}$	$M_{2,1}$	$2M_{2,1}$	$M_{1,1}$
$M_{1,2}$	$M_{1,2}$	$M_{2,2}$	M_2	$M_{2,1}$	M_4	$M_{1,1}$
M_4	M_4	$M_{2,2}$	$M_{2,2}$	$2M_4$	M_4	$2M_{2,2}$
$M_{2,1}$	$M_{2,1}$	$M_{1,1}$	$M_{1,1}$	$2M_{2,1}$	$M_{2,1}$	$2M_{2,2}$
$M_{2,2}$	$M_{2,2}$	$2M_{2,2}$	M_4	M_4	$2M_4$	$M_{2,2}$

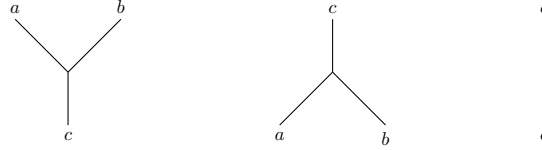


Figure 13 Left: V_c^{ab} ; Middle: V_{ab}^c ; Right: id_c

$M_{2,1}$ and $M_{2,2}$ corresponding to $A \times 0$ and $0 \times A$ are actually the $D(\mathbb{Z}_2)$ -modules which were made into bimodules using the braiding structure. Therefore, let us calculate the bimodule products $M_{2,i}^2$. We start with $i = 2$. Recall that the bicharacter is always

$$\psi((x_1, y_1), (x_2, y_2)) = e^{\frac{2\pi i}{2N}(y_{22}(x_{11}+y_{11})-(x_{21}+y_{21})y_{12})}.$$

The algebra B is $x \oplus y \in \mathbb{Z}_2^2 \oplus \mathbb{Z}_2^2$ such that $x + y \in 0 \times \mathbb{Z}_2$. This means it has 16 elements

$$(0 \times \mathbb{Z}_2, 0 \times \mathbb{Z}_2) \cup (1 \times \mathbb{Z}_2, 1 \times \mathbb{Z}_2). \tag{A.5}$$

On B , the bicharacter is trivial as $x_{11} + y_{11} = x_{21} + y_{21} = 0$. To compute $\mathcal{M}(B, \psi_{\text{trivial}}) \boxtimes_{\text{Vec}_{\mathbb{Z}_2^2}} \mathcal{M}(B, \psi_{\text{trivial}})$, we look at the intersection of $B \cap B$ inside the second component of $\mathbb{Z}_2^2 \oplus \mathbb{Z}_2^2$ and it is $0 \times \mathbb{Z}_2$. The number of elements inside $B \circ B$ is exactly 16, with elements given by

$$(0 \times \mathbb{Z}_2, a, a, 0 \times \mathbb{Z}_2) \cup (1 \times \mathbb{Z}_2, b, b, 1 \times \mathbb{Z}_2), \quad \text{where } a \in 0 \times \mathbb{Z}_2, \quad b \in 1 \times \mathbb{Z}_2.$$

This is equal to $(B \cap B)^\perp$ as the character is trivial. Hence, the multiplicity formula from the proposition gives

$$m = \frac{2 \cdot 16 \cdot 4}{8 \cdot 8} = 2 \tag{A.6}$$

and the algebra H'' is the image of $(B \cap B)^\perp = B \circ B$ in $\mathbb{Z}_2^2 \oplus \mathbb{Z}_2^2$ when we forget the middle two elements, which means we get B back. Hence, $M_{2,2} \boxtimes_{D\mathbb{Z}_2} M_{2,2} = 2M_{2,2}$.

The procedure is quite similar for the case $M_{2,1}$, where the only subtlety is when we want to compute the orthogonal complement $(B \cap B)^\perp$ as the bicharacter ψ is no longer trivial. We have $B \cap B = \{(0, a, a, 0), a \in \mathbb{Z}_2 \times 0\}$ and we have to find all $(x, y, y, t) \in B \circ B$ such that $\psi((0, a), (x, y))\psi((a, 0), (y, t)) = 1$ for any element in $B \cap B$. Recall that $B \circ B = \{(x, y, y, t) \mid x + y, y + t \in A = \mathbb{Z}_2 \times 0\}$. It can be seen easily that for $a = (1, 0)$ and any element (x, y, y, t) with $x, y, t \in \mathbb{Z}_2 \times 1$ the above will be -1 . In fact, $\psi((0, a), (x, y))\psi((a, 0), (y, t)) = e^{\frac{\pi i}{2}(y_2+t_2)a_1}$ and otherwise, when $x, y, t \in \mathbb{Z}_2 \times 0$, it is equal to one. Therefore the orthogonal complement has 8 elements which are $x, y, t \in \mathbb{Z}_2 \times 0$. Hence the multiplicity can be seen to be 1. The algebra of the result $\mathcal{M}(H, \xi) = M_{2,1} \boxtimes_{D\mathbb{Z}_2} M_{2,1}$ will be the image of the orthogonal complement as we drop the middle two components which is $H = (\mathbb{Z}_2 \times 0, \mathbb{Z}_2 \times 0)$. As for its bicharacter ξ , it must be the image of $\psi \times \psi$ under the projection, and the way to compute it is to take any preimage and compute $\psi \times \psi$. It can be seen to be trivial by taking preimages $(x, 0, 0, t), (x', 0, 0, t') \in (B \cap B)^\perp$ where $(x, t), (x', t') \in H \Rightarrow x, t, x', t' \in \mathbb{Z}_2 \times 0$. We compute $\psi((x, 0), (x', 0))\psi((0, t), (0, t'))$ and both are 1. All in all

$$M_{2,2} \boxtimes_{D\mathbb{Z}_2} M_{2,2} = 2M_{2,2}, \tag{A.7}$$

$$M_{2,1} \boxtimes_{D\mathbb{Z}_2} M_{2,1} = \mathcal{M}((\mathbb{Z}_2 \times 0, \mathbb{Z}_2 \times 0), \psi_{\text{trivial}}). \tag{A.8}$$

A.2 HM on $2 + \tau$

We present a Hopf algebra structure on $2 + \tau$ in the Fibonacci category Fib .

Throughout the section, $\phi = \frac{\sqrt{5}+1}{2}$ is the golden ratio. Some other notations are in order. For any simple objects a, b and c such that c is a subobject of $a \otimes b$, we arbitrarily choose and fix $V_c^{ab} \in \text{Hom}(c, a \otimes b)$, $V_{ab}^c \in \text{Hom}(a \otimes b, c)$ such that $V_{ab}^c \circ V_c^{ab} = \text{id}_c$. The graphical representations for V_c^{ab} , V_{ab}^c

and id_c are given in Figure 13. For an object $X = \sum_a X_a a$ where X_a represents the multiplicities of a , denote by a_i the i -th copy of a in X , $i = 1, \dots, X_a$. If $X_a = 1$, we simply write a instead of a_1 . Then a morphism f from $X \otimes Y$ to Z is fully specified by

$$f = \sum_{a,b,c} \sum_{i,j,k} f_{abc}^{ijk} \begin{array}{c} c_k \\ \diagup \quad \diagdown \\ a_i \quad b_j \end{array}$$

for some scalars f_{abc}^{ijk} . Usually, the terms with vanishing f_{abc}^{ijk} will be dropped out from the above equation. Morphisms from Z to $X \otimes Y$, or from X to Y are specified in a similar way in terms of V_c^{ab} 's or id_c 's.

With assistance of Mathematica, we conclude that there is a unique Hopf algebra structure on $2 + \tau$. Here is a brief summary. The Hopf algebra is not commutative, nor cocommutative. It has a two sided integral on which the value of the counit is not zero. Hence it is semisimple. However, the antipode has order 10. Thus, semisimplicity of Hopf algebras in Fib is not equivalent to the condition that the antipode has order two, unlike complex finite dimensional Hopf algebras in $\text{Vec}_{\mathbb{C}}$. $2 + \tau$ has four irreducible modules whose underlying objects are given by $\mathbf{1}, \tau, \tau$ and $\mathbf{1} \oplus \tau$. These modules form the category DFib. We present the Hopf algebra structure maps as well as the module structures below.

The structure maps $M, \Delta, \eta, \epsilon, S$ on $2 + \tau$ are defined, respectively, in (A.9)–(A.13), where in (A.10), we define

$$V = \left(-\frac{\phi^{-2}}{2} + \frac{5^{\frac{1}{4}}\phi^{-\frac{1}{2}}}{2}i \right), \quad U = \frac{i\sqrt[4]{5(-9 + 4\sqrt{5} - i\sqrt{1525 - 682\sqrt{5}})}}{2^{3/4}},$$

$$\begin{array}{c} \begin{array}{ccccc} \mathbf{1}_1 & & \mathbf{1}_2 & & \mathbf{1}_2 & & \mathbf{1}_1 & & \mathbf{1}_1 \\ | & & | & & | & & | & & | \\ \diagdown & & \diagdown & & \diagdown & & \diagdown & & \diagdown \\ \mathbf{1}_1 & & \mathbf{1}_1 & & \mathbf{1}_2 & & \mathbf{1}_2 & & \tau \\ \diagup & & \diagup & & \diagup & & \diagup & & \diagup \\ \tau & & \tau & & \tau & & \tau & & \tau \end{array} + \dots \\ - \sqrt{2}\phi^{-\frac{3}{4}} \begin{array}{c} \tau \\ | \\ \diagdown \quad \diagup \\ \tau \quad \tau \end{array}, \end{array} \tag{A.9}$$

$$\begin{array}{c} \begin{array}{cccc} \mathbf{1}_1 & \mathbf{1}_1 & \mathbf{1}_1 & \mathbf{1}_1 \\ | & | & | & | \\ \diagdown & \diagdown & \diagdown & \diagdown \\ \mathbf{1}_1 & \mathbf{1}_2 & \mathbf{1}_2 & \mathbf{1}_2 \\ \diagup & \diagup & \diagup & \diagup \\ \mathbf{1}_2 & \mathbf{1}_2 & \tau & \mathbf{1}_2 \\ \diagdown & \diagdown & \diagdown & \diagdown \\ \mathbf{1}_2 & \tau & \tau & \tau \end{array} - \frac{1}{2\phi} \dots + \frac{1}{2\phi} \dots + \frac{1}{2\phi} \dots \\ + \frac{\sqrt{5}}{2\phi} \dots + V \dots + \frac{1}{2\phi} \dots + \frac{\sqrt{5}}{2\phi} \dots \end{array}$$

$$\begin{aligned}
 & + \frac{1}{2\phi} \begin{array}{c} \tau \quad \mathbf{1}_1 \\ \diagdown \quad / \\ \text{Y} \\ / \quad \backslash \\ \tau \end{array} + \frac{\sqrt{5}}{2\phi} \begin{array}{c} \tau \quad \mathbf{1}_2 \\ \diagdown \quad / \\ \text{Y} \\ / \quad \backslash \\ \tau \end{array} - U \begin{array}{c} \tau \quad \tau \\ \diagdown \quad / \\ \text{Y} \\ / \quad \backslash \\ \tau \end{array}, \tag{A.10}
 \end{aligned}$$

$$\begin{array}{c} \mathbf{1}_1 \\ | \\ \mathbf{1} \end{array}, \tag{A.11}$$

$$\begin{array}{c} \mathbf{1} \\ | \\ \mathbf{1}_1 \end{array} + \begin{array}{c} \mathbf{1} \\ | \\ \mathbf{1}_2 \end{array}, \tag{A.12}$$

$$\begin{array}{c} \mathbf{1}_1 \\ | \\ \mathbf{1}_1 \end{array} + \begin{array}{c} \mathbf{1}_2 \\ | \\ \mathbf{1}_2 \end{array} - \left(\frac{\phi^{-1}}{2} + \frac{5^{\frac{1}{4}}\phi^{\frac{1}{2}}}{2}i \right) \begin{array}{c} \tau \\ | \\ \tau \end{array}. \tag{A.13}$$

$2 + \tau$ has four equivalent classes of irreducible modules. The trivial module is given by $M_0 = (\mathbf{1}, \epsilon)$. There are two module structures on τ which we denote by $M_i = (\tau, r_i)$, $i = 1, 2$. There is also a module on $\mathbf{1} + \tau$ denoted by $M_3 = (\mathbf{1} + \tau, r_3)$. The action r_i giving the module structure for $i = 1, 2, 3$ is presented, respectively, in (A.14)–(A.16). With the data of $2 + \tau$ and its irreducible modules, it is straightforward to check (in Mathematica) that the category of $(2 + \tau)$ -modules is isomorphic to DFib. We have

$$\begin{array}{c} \tau \\ | \\ \text{Y} \\ / \quad \backslash \\ \mathbf{1}_1 \quad \tau \end{array} + \begin{array}{c} \tau \\ | \\ \text{Y} \\ / \quad \backslash \\ \mathbf{1}_2 \quad \tau \end{array}, \tag{A.14}$$

$$\begin{array}{c} \tau \\ | \\ \text{Y} \\ / \quad \backslash \\ \mathbf{1}_1 \quad \tau \end{array} - \begin{array}{c} \tau \\ | \\ \text{Y} \\ / \quad \backslash \\ \mathbf{1}_2 \quad \tau \end{array} + \sqrt{2}\phi^{\frac{1}{4}} \begin{array}{c} \tau \\ | \\ \text{Y} \\ / \quad \backslash \\ \tau \quad \tau \end{array}, \tag{A.15}$$

$$\begin{aligned}
 & \begin{array}{c} \mathbf{1} \\ | \\ \text{Y} \\ / \quad \backslash \\ \mathbf{1}_1 \quad \mathbf{1} \end{array} - \begin{array}{c} \mathbf{1} \\ | \\ \text{Y} \\ / \quad \backslash \\ \mathbf{1}_2 \quad \mathbf{1} \end{array} + \begin{array}{c} \mathbf{1} \\ | \\ \text{Y} \\ / \quad \backslash \\ \tau \quad \tau \end{array} + \begin{array}{c} \tau \\ | \\ \text{Y} \\ / \quad \backslash \\ \mathbf{1}_1 \quad \tau \end{array} \\
 & - \begin{array}{c} \tau \\ | \\ \text{Y} \\ / \quad \backslash \\ \mathbf{1}_2 \quad \tau \end{array} + 2 \begin{array}{c} \tau \\ | \\ \text{Y} \\ / \quad \backslash \\ \tau \quad \mathbf{1} \end{array} - \sqrt[4]{4\sqrt{5}-8} \begin{array}{c} \tau \\ | \\ \text{Y} \\ / \quad \backslash \\ \tau \quad \tau \end{array}. \tag{A.16}
 \end{aligned}$$

A.3 A condensible algebra structure on $A + C$ in $\text{Rep}(S_3) \subset D(S_3)$

Here, we give explicitly the condensible algebra structure on $\mathcal{A} = A \oplus C$ in $\text{Rep}(S_3) \subset D(S_3)$. We refer the reader to Appendix A of [13] for the F -symbols and R -symbols of $D(S_3)$. A particularly important

F -matrix used here is

$$F_C^{CCC} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

With the graphical notions as in Subsection A.2, the multiplication M on \mathcal{A} is given in Equation (A.17) and the unit map $\eta: A \rightarrow \mathcal{A}$ is the apparent embedding. \mathcal{A} is commutative. The multiplication M has a section $r: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ which is defined in Equation (A.18). It is straightforward to check that r is a bi-module map. Hence \mathcal{A} is separable.

$$\begin{array}{c}
 \begin{array}{c} A \\ | \\ \diagdown \quad \diagup \\ A \quad A \end{array} + \sqrt{2} \begin{array}{c} A \\ | \\ \diagdown \quad \diagup \\ C \quad C \end{array} + \begin{array}{c} C \\ | \\ \diagdown \quad \diagup \\ A \quad C \end{array} + \begin{array}{c} C \\ | \\ \diagdown \quad \diagup \\ C \quad A \end{array} + \begin{array}{c} C \\ | \\ \diagdown \quad \diagup \\ C \quad C \end{array}, \tag{A.17}
 \end{array}$$

$$\begin{array}{c}
 \frac{1}{3} \begin{array}{c} A \quad A \\ \diagdown \quad \diagup \\ A \end{array} + \frac{\sqrt{2}}{3} \begin{array}{c} C \quad C \\ \diagdown \quad \diagup \\ A \end{array} + \frac{1}{3} \begin{array}{c} A \quad C \\ \diagdown \quad \diagup \\ C \end{array} + \frac{1}{3} \begin{array}{c} C \quad A \\ \diagdown \quad \diagup \\ C \end{array} + \frac{1}{3} \begin{array}{c} C \quad C \\ \diagdown \quad \diagup \\ C \end{array}. \tag{A.18}
 \end{array}$$

A.4 Fusion rules of $(\text{SO}(8)_1)^{\times, S_3}$

Here, we list the fusion rules of Example 5.2.3 as provided in [12, Appendix B] in our new notation; it should be noted the two theories in [12, Appendix B] are the same. We will use \pm and \mp to shorten the list. The only rule is that if a choice is made for the first (second) superscript then the corresponding choice must be made throughout the equation for the first (second) superscript; e.g., in $X^{\pm\pm} \otimes \dots = X^{\pm\mp} \oplus \dots$, we can choose $X^{+-} \otimes \dots = X^{++} \oplus \dots$.

- $B \otimes B = A$;
- $B \otimes C = C$;
- $B \otimes Y_{\pm} = Y_{\mp}$;
- $B \otimes X = X$;
- $B \otimes {}_{\alpha}X = {}_{\alpha}X$;
- $B \otimes {}_{\alpha^*}X = {}_{\alpha^*}X$;
- $B \otimes X^{\pm\pm} = X^{\pm\mp}$;
- $C \otimes C = A \oplus B \oplus C$;
- $C \otimes Y_{\pm} = Y_+ \oplus Y_-$;
- $C \otimes X = {}_{\alpha}X \oplus {}_{\alpha^*}X$;
- $C \otimes {}_{\alpha}X = X \oplus {}_{\alpha^*}X$;
- $C \otimes {}_{\alpha^*}X = X \oplus {}_{\alpha}X$;
- $C \otimes X^{\pm\pm} = X^{\pm\pm} \oplus X^{\pm\mp}$;
- $Y_{\pm} \otimes Y_{\pm} = A \oplus C \oplus Y_+ \oplus Y_-$;
- $Y_+ \otimes Y_- = B \oplus C \oplus Y_+ \oplus Y_-$;
- $Y_{\pm} \otimes Z = X \oplus {}_{\alpha}X \oplus {}_{\alpha^*}X, \forall Z \in \{X, {}_{\alpha}X, {}_{\alpha^*}X\}$;
- $X \otimes X = A \oplus B \oplus Y_+ \oplus Y_- \oplus X \oplus {}_{\alpha^*}X$;
- $X \otimes {}_{\alpha}X = C \oplus Y_+ \oplus Y_- \oplus {}_{\alpha}X \oplus {}_{\alpha^*}X$;
- $X \otimes {}_{\alpha^*}X = C \oplus Y_+ \oplus Y_- \oplus X \oplus {}_{\alpha}X$;
- $Z \otimes X^{\pm\pm} = \oplus_{i,j \in \{\pm 1\}} X^{ij}, \forall Z \in \{X, {}_{\alpha}X, {}_{\alpha^*}X\}$;
- ${}_{\alpha}X \otimes {}_{\alpha}X = A \oplus B \oplus Y_+ \oplus Y_- \oplus X \oplus {}_{\alpha}X$;
- ${}_{\alpha}X \otimes {}_{\alpha^*}X = C \oplus Y_+ \oplus Y_- \oplus X \oplus {}_{\alpha^*}X$;
- ${}_{\alpha^*}X \otimes {}_{\alpha^*}X = A \oplus B \oplus Y_+ \oplus Y_- \oplus {}_{\alpha}X \oplus {}_{\alpha^*}X$;

• $X^{\pm\pm} \otimes X^{\mp\pm} = Y_+ \oplus Y_- \oplus X \oplus {}_{\alpha}X \oplus {}_{\alpha^*}X$, exceptionally in this equation the *second* superscripts can be chosen independently;

• $Y_+ \otimes X^{++} = X^{++} \oplus X^{-+} \oplus X^{--};$

• $Y_+ \otimes X^{+-} = X^{+-} \oplus X^{-+} \oplus X^{--};$

• $Y_+ \otimes X^{-+} = X^{++} \oplus X^{+-} \oplus X^{--};$

• $Y_+ \otimes X^{--} = X^{++} \oplus X^{+-} \oplus X^{-+};$

• $Y_- \otimes X^{++} = X^{+-} \oplus X^{-+} \oplus X^{--};$

• $Y_- \otimes X^{+-} = X^{++} \oplus X^{-+} \oplus X^{--};$

• $Y_- \otimes X^{-+} = X^{++} \oplus X^{+-} \oplus X^{--};$

• $Y_- \otimes X^{--} = X^{++} \oplus X^{+-} \oplus X^{-+};$

• $X^{\pm\pm} \otimes X^{\pm\pm} = A \oplus C \oplus Y_{\pm} \oplus X \oplus {}_{\alpha}X \oplus {}_{\alpha^*}X$, where the same choice is for the first subscript and first superscript;

• $X^{++} \otimes X^{+-} = B \oplus C \oplus Y_- \oplus X \oplus {}_{\alpha}X \oplus {}_{\alpha^*}X;$

• $X^{-+} \otimes X^{--} = B \oplus C \oplus Y_+ \oplus X \oplus {}_{\alpha}X \oplus {}_{\alpha^*}X.$