# CONSTRUCTING APPROXIMATELY DIAGONAL UNITARY GATES 

COLTON GRIFFIN AND SHAWN X. CUI


#### Abstract

We study a method of producing approximately diagonal 1-qubit gates. For each positive integer, the method provides a sequence of gates that are defined iteratively from a fixed diagonal gate and an arbitrary gate. These sequences are conjectured to converge to diagonal gates doubly exponentially fast and are verified for small integers. We systemically study this conjecture and prove several important partial results. Some techniques are developed to pave the way for a final resolution of the conjecture. The sequences provided here have applications in quantum search algorithms, quantum circuit compilation, generation of leakage-free entangled gates in topological quantum computing, etc.


## 1. Introduction

A basic question in quantum computing is to approximate an arbitrary quantum gate efficiently with a universal gate set. The Solovay-Kitaev theorem provides a general solution to this question. Given a universal gate set in $\mathrm{SU}(d)$, the theorem provides an algorithm to approximate an arbitrary gate of $\mathrm{SU}(d)$ with running time and space complexity both $O\left(\log ^{c}(1 / \epsilon)\right)$ to an accuracy $\epsilon>0$ [4]. Here $c \approx 3$ with its explicit value varying depending on the realizations of the theorem. However, this algorithm is usually not optimal and more efficient approximation protocols exist on certain specific gate sets. Developing optimal approximation protocols is especially critical for systems that have potential experimental implementations. One such example is the Fibonacci anyon circuit, one of the most prominent models for topological quantum computing [5]. In this model, there are algorithms for approximation where the exponent $c$ can be improved to be asymptotically optimal $c=1[7,8]$.

We focus on the methods used in [8] where a key tool to obtain the optimal $c=1$ is the following proposition. Let $D(\theta)=\operatorname{diag}\left(1, e^{i \theta}\right)$. For any unitary $U_{0} \in \mathrm{SU}(2)$ and $\theta=\frac{\pi}{5}$, consider the recursive sequence,

$$
\begin{equation*}
U_{k+1}=U_{k} D(\theta) U_{k}^{-1} D(\theta)^{3} U_{k} D(\theta)^{3} U_{k}^{-1} D(\theta) U_{k} \tag{1}
\end{equation*}
$$

In was shown that $\left|\left(U_{k+1}\right)_{12}\right|=\left|\left(U_{k}\right)_{12}\right|^{5}$ where $U_{i j}$ denotes the $(i, j)$-entry of $U[8,9]$. Hence, if $U_{0}$ is not diagonal, the sequence in Equation 1 converges to a diagonal gate ${ }^{1}$. The convergence is double exponentially fast and the space complexity is $O(\log (1 / \epsilon))$ to reach the limit within precision $\epsilon>0$. The above technique is also heavily utilized to design composite pulse sequence for quantum error correction [9] and to generate leakage-free entangling 2 -qubit gates in the Fibonacci model $[2,3]$.

Equation 1 in turn is inspired by a simpler one from Grover's quantum search algorithm [6]. That is, for $\theta=\frac{\pi}{3}$, consider instead the sequence,

$$
\begin{equation*}
U_{k+1}=U_{k} D(\theta) U_{k}^{-1} D(\theta) U_{k} \tag{2}
\end{equation*}
$$

Then it is straightforward to check that $\left|\left(U_{k+1}\right)_{12}\right|=\left|\left(U_{k}\right)_{12}\right|^{3}$. Besides Grover's search algorithm, this sequence is also used in some other quantum algorithms $[11,12]$.

The relation between the ( 1,2 )-entry (and also the ( 2,1 )-entry) of adjacent terms in the above two sequences is intriguing as it is an exact equality. This motivates the question of whether they are special cases of a more general pattern. That is, for each odd integer $p>1$ enumerated by $p=2 n+1$ for $n \in \mathbb{N}$ and $\theta_{p}=\frac{\pi}{p}$, is there an recursive sequence $\left\{U_{k}^{(n)}\right\}_{k=0}^{\infty}$, defined similar in form to those in Equations 1 and 2 with $D(\theta)$ replaced by $D\left(\theta_{p}\right)$ such that $\left|\left(U_{k+1}^{(n)}\right)_{12}\right|=\left|\left(U_{k}^{(n)}\right)_{12}\right|^{2 n+1}=\left|\left(U_{k}^{(n)}\right)_{12}\right|^{p}$ ? Such a generalization not only is interesting on its own as a mathematical proposition, but also has applications in topological quantum computing. Recall that the Fibonacci model is described by the Witten-Chen-Simons theory $\mathrm{SU}(2)_{3}$ in which

[^0]braiding of anyons naturally gives the diagonal gate $D\left(\frac{\pi}{5}\right)$ (ref. [3]) and hence Equation 1 can be used in this model. The theory $\mathrm{SU}(2)_{p}$ is also defined for any $p \geq 1$, and for odd $p$, braiding of anyons gives the diagonal gate $D\left(\frac{\pi}{p+2}\right)$. This can be obtained from the $R$-symbols of the theory (Ref. [1], Sec. 5.4). Therefore, the generalized sequence $\left\{U_{k}^{(n)}\right\}_{k=0}^{\infty}$ will be useful in the $\mathrm{SU}(2)_{2 n-1}$ anyon model for both topological compilation and generation of entangled gates.

A conjectured formula for the generalized sequence was given in [8] for each odd $p$ (see also Section 2.1 for an explicit form). For each $p=2 n+1$, the sequence $\left\{U_{k}^{(n)}\right\}$ is defined in a recursive formula similar to those in Equations 1 and 2. The length of the words in the recursion is $O(n)$. Conjecture 2.1 states that $\left|\left(U_{k+1}^{(n)}\right)_{12}\right|=\left|\left(U_{k}^{(n)}\right)_{12}\right|^{2 n+1}$.

In the current paper, we systemically study this conjecture. For each odd $p=2 n+1>1$, we analyze the entries of $U_{k+1}^{(n)}$ in terms of those of $U_{k}^{(n)}$ and present them in a specially designed form. Explicitly, let $a_{k}=\left(U_{k}^{(n)}\right)_{11}$ and $b_{k}=\left(U_{k}^{(n)}\right)_{12}$ (ignoring the dependence of $a_{k}$ and $b_{k}$ on $n$ for now). Then

$$
\begin{equation*}
\frac{b_{k+1}}{b_{k}}=\beta_{0}+\left|a_{k}\right|^{2}\left(\beta_{1}-\left|b_{k}\right|^{2}\left(\beta_{2}+\left|a_{k}\right|^{2}\left(\beta_{3}-\left|b_{k}\right|^{2}(\ldots)\right)\right)\right) \tag{3}
\end{equation*}
$$

where $\beta_{0}, \cdots, \beta_{n}$ are expressions involving entries of $D\left(\theta_{p}\right)$. By using certain induction technique, we provide an explicit formula for the $\beta_{j}{ }^{\prime}$ s (Theorem 5.7). In general, these $\beta_{j}$ 's are very complicated and hence difficult to evaluate. However, we are able to compute the values for $\beta_{0}, \beta_{1}$, and $\beta_{n}$. Furthermore, we give a conjecture for the values of all $\beta_{j}$ 's (Corollary 4.4). The derivation of such values itself is quite non-trivial and involves technical identities of binomial coefficients. A complete verification for the conjectured values would lead to a proof of Conjecture 2.1. This is left for a future direction. As a concrete application, we prove Conjecture 2.1 for $p=7$, with the corresponding sequence given by,

$$
\begin{equation*}
U_{k+1}=U_{k} D(\theta) U_{k}^{-1} D(\theta)^{5} U_{k} D(\theta)^{3} U_{k}^{-1} D(\theta)^{3} U_{k} D(\theta)^{5} U_{k}^{-1} D(\theta) U_{k} \tag{4}
\end{equation*}
$$

We also show that there is an inductive formula to obtain the sequence for $p+2$ from that for $p$. This provides a potential approach to proving the conjecture using induction on $p$. In addition, we show that two sequences for $p_{1}$ and $p_{2}$ respectively can be combined to obtain a sequence for $p_{1} p_{2}$ which is different from the one constructed from the conjecture. As a corollary, there exists a sequence for $p=15=3 \cdot 5$ with the desired property but different from the conjectured sequence for $p=15$.

The rest of the paper is organized as follows. In Section 2, we provide some backgrounds and define the sequence for each odd $p$ following [8]. In Section 3, we present the matrix entries in each sequence in a special form and show how adjacent sequences are related with each other. Sections 4 and 5 are devoted to studying the special form to greater details, including deriving explicit formulas for each term and evaluating some of the formulas. Section 6 provides an alternative approach to obtaining the sequence for a composite integer from those of its prime factors.

## 2. Preliminaries

We begin with some notation. We use the following construction of a unitary matrix $U_{k} \in \mathrm{U}(2)$ :

$$
U_{k}=e^{i \varphi_{k} / 2}\left(\begin{array}{cc}
a_{k} & -\overline{b_{k}}  \tag{5}\\
b_{k} & \overline{a_{k}}
\end{array}\right)
$$

where $\left|a_{k}\right|^{2}+\left|b_{k}\right|^{2}=1$ and $\varphi_{k} \in \mathbb{R}$, hence $\operatorname{det} U=e^{i \varphi_{k}}$. If $\varphi_{k}=0$, then $U \in \operatorname{SU}(2)$, so this definition doubles as a construction for the special unitary matrices, which we will also make use of. In this notation, the upper left element of $U_{k}$ is $\left(U_{k}\right)_{11} \equiv e^{i \varphi_{k} / 2} a_{k}$ and the lower left element of $U_{k}$ is $\left(U_{k}\right)_{21} \equiv e^{i \varphi_{k} / 2} b_{k}$. This extra phase will not be important, since throughout most of the paper we will let $\varphi_{k}=0$. Hence the upper and lower left elements of $U_{k}$ will be referred to as $a_{k}$ and $b_{k}$. We also denote $\chi_{j} \equiv j \bmod 2$ since this will be used often throughout the paper.
2.1. Diagonalizing Sequences. As before, fix an odd integer $p \equiv 2 n+1$ for $n \in \mathbb{N}$ as well as an input unitary matrix $U_{0} \in \mathrm{U}(2)$. We wish to construct a sequence $\left\{U_{k}^{(n)}\right\}_{k=0}^{\infty}$ defined recursively from any $U_{0}$, such that $U_{k+1}$ is expressed as a product of matrices $U_{k}, U_{k}^{-1}$, and a set of conditional phase shift gates to be chosen. These sequences should have the property that for each $k$, we get an optimal convergence rate $\left|\left(U_{k+1}^{(n)}\right)_{21}\right| \equiv\left|b_{k+1}^{(n)}\right|=\left|b_{k}^{(n)}\right|^{2 n+1}$, which would mean that unless $\left|b_{0}\right|=1$, we get that $\left\{U_{k}^{(n)}\right\}$ converges quickly to a diagonal matrix. The ( $n$ ) notation here on the sequences denotes which integer $n$ the sequence
is converging with respect to. We will also refer to $n$ as the order of the sequence $\left\{U_{k}^{(n)}\right\}_{k=0}^{\infty}$. If the order $n$ referred to is clear enough in context it will sometimes be dropped. The objects which will have this notation applied are the elements of our sequences $U_{k}^{(n)}$ and their sub-elements, such as $\left(U_{k}^{(n)}\right)_{11} \equiv e^{i \varphi_{k} / 2} a_{k}^{(n)}$. Sometimes powers or inverses will be applied to various functions, but these will be written without a parenthesis around the superscript.

Now let us explicitly define these sequences. Let $D_{j}\left(\theta_{p}\right)$ be diagonal matrices indexed by $j$, with $\theta_{p} \equiv$ $\frac{\pi}{p} \equiv \frac{\pi}{2 n+1}$. These matrices will be defined as $D_{j}\left(\theta_{p}\right) \equiv \operatorname{diag}\left(\lambda_{j 0}, \lambda_{j 1}\right)$, where $\lambda_{j 0}$ and $\lambda_{j 1}$ (we will also refer to these pairs of elements as $\lambda_{j l}$, where $l=0,1$ ) are defined as the roots of unity

$$
\begin{equation*}
\lambda_{j 0}=\omega^{(-1)^{j} j}, \quad \lambda_{j 1}=(-1)^{j+1} \omega^{(-1)^{j+1} j} \tag{6}
\end{equation*}
$$

for $\omega=e^{i \theta_{p} / 2}$, and again $j=1, \ldots, n$. We let our diagonalizing sequences be defined by the recursive equation $U_{k+1}^{(n)} \equiv A_{p}\left(U_{k} ; \theta_{p}\right) \equiv Q_{n} U_{k}^{(-1)^{n}} P_{n}$, where $P_{n}$ and $Q_{n}$ are defined recursively as

$$
\begin{align*}
P_{j+1} & =D_{j+1}\left(\theta_{p}\right) U_{k}^{(-1)^{j}} P_{j} \\
Q_{j+1} & =Q_{j} U_{k}^{(-1)^{j}} D_{j+1}\left(\theta_{p}\right) \tag{7}
\end{align*}
$$

where $P_{0}=Q_{0}=I$. Here we are using Reichardt's notation from [8, Eq. 6] for generating these sequences. Hence for a given $n$, we have that $U_{k+1}^{(n)}$ is expressed as a product of $U_{k}, U_{k}^{-1}$, and $D_{j}\left(\theta_{p}\right)$ for $j=1, \ldots, n$. For example, the first two diagonalizing sequences $\left\{U_{k}^{(1)}\right\}$ and $\left\{U_{k}^{(2)}\right\}$ are given by the recursive equations

$$
U_{k+1}^{(1)}=U_{k} D_{1} U_{k}^{-1} D_{1} U_{k}, \quad U_{k+1}^{(2)}=U_{k} D_{1} U_{k}^{-1} D_{2} U_{k} D_{2} U_{k}^{-1} D_{1} U_{k}
$$

In Reichardt's definition the above sequences, he used the definition $D_{j}^{\prime}\left(\theta_{p}\right) \equiv \operatorname{diag}\left(1,(-1)^{j+1} e^{i \theta_{p}(-1)^{j+1} j}\right)$ for the diagonal matrices, which correspond to certain conditional phase shifts. Our definition for $D_{j}\left(\theta_{p}\right)$ is almost the exact same; in fact if we write $D_{j}^{\prime}\left(\theta_{p}\right) \equiv \operatorname{diag}\left(1,(-1)^{j+1} e^{i \theta_{p}(-1)^{j+1} j}\right)$, then $D_{j}^{\prime}\left(\theta_{p}\right)=\lambda_{j 0}^{-1} D_{j}\left(\theta_{p}\right)$. The benefit of using this definition is that $\lambda_{j 0} \lambda_{j 1}=(-1)^{j+1}$ and $\overline{\lambda_{j 0}^{2}}=\lambda_{j 1}^{2}$, both of which are very useful identities as we shall see. The conjectured property of these sequences is the following:

Conjecture 2.1. Given any $U_{0} \in \mathrm{U}(2)$, the diagonalizing sequences $\left\{U_{k}^{(n)}\right\}_{k=0}^{\infty}$ defined by the recursive equation $U_{k+1}^{(n)} \equiv A_{p}\left(U_{k} ; \theta_{p}\right) \equiv Q_{n} U_{k}^{(-1)^{n}} P_{n}$ have the property that

$$
\begin{equation*}
\left|\left(U_{k+1}^{(n)}\right)_{21}\right| \equiv\left|b_{k+1}^{(n)}\right|=\left|b_{k}^{(n)}\right|^{2 n+1} \equiv\left|b_{k}^{(n)}\right|^{p} \tag{8}
\end{equation*}
$$

Before we begin the analysis, it is worth noting a few properties of these types of sequences. The definition of $A_{p}$ will be used in Section 6 to refer to any sequence that has the conjectured property above but sparingly otherwise in order to avoid mixed notation. As mentioned before, as $k \rightarrow \infty$ the matrices $U_{k}^{(n)}$ converge to a diagonal unitary matrix with one exception. The sequences defined above preserve unitarity for each $k$ by group closure since $D_{j}$ is unitary for any $n$ and $j=1, \ldots, n$. Thus $\left|a_{k}^{(n)}\right|^{2}+\left|b_{k}^{(n)}\right|^{2}=1$ for all $k$, and so $\left|b_{k}^{(n)}\right| \leq 1$, where again $\left(U_{k}^{(n)}\right)_{11} \equiv e^{i \varphi_{k} / 2} a_{k}^{(n)}$ and $\left(U_{k}^{(n)}\right)_{21} \equiv e^{i \varphi_{k} / 2} b_{k}^{(n)}$. Unless $\left|b_{k}^{(n)}\right|=1$ for some $k$, these sequences will converge to a diagonal unitary matrix. In fact, $\left|b_{k}^{(n)}\right|=1$ for some $k$ implies $\left|b_{0}^{(n)}\right|=1$ by the conjecture.

It is also worth noting that our choice for $\lambda_{j l}$ is not unique at least up to a factor $e^{i \gamma}$ in the sense that for a sequence defined by the recursion $U_{k+1}^{(n)} \equiv e^{i \gamma} A_{p}\left(U_{k} ; \theta_{p}\right)$, if the sequence has the property $\left|\left(U_{k+1}^{(n)}\right)_{21}\right|=$ $\left|\left(U_{k}^{(n)}\right)_{21}\right|^{p}$ for a specific choice of $\gamma$, then the property will hold for any $\gamma$ since it does not affect the magnitude. For example, we can pull the $\lambda_{j 0}$ factor out for each matrix $D_{j}$ and obtain the diagonals $D_{j}^{\prime}=\operatorname{diag}\left(1,(-1)^{j+1} e^{i \theta_{p}(-1)^{j+1} j}\right)$, the same definition as given in the Reichardt paper. Then we get that $U_{k+1}^{(n)}=\left(\prod_{j=1}^{n} \lambda_{j 0}^{-2}\right) A_{p}\left(U_{k} ; \theta_{p}\right)$ yields the same sequence as if we replaced $D_{j}\left(\theta_{p}\right)$ with the diagonals $D_{j}^{\prime}\left(\theta_{p}\right)$ that Reichardt used. Then if the sequences optimally converge using the diagonals $D_{j}$, the same sequences converge if we instead use $D_{j}^{\prime}$ in their place. That said, this is our choice for $\lambda_{j l}$ due to some of its convenient properties, and so when we refer to $\lambda_{j 0}$ and $\lambda_{j 1}$, we mean the definitions in (6) unless stated otherwise.

## 3. Inductively Constructing Sequences of Arbitrary Order

In order to analyze these sequences in generality, we apply an inductive argument to represent $U_{k+1}^{(n+1)}$ in terms of $U_{k}^{(n)}$. Without loss of generality, it suffices to prove our desired property for $U_{0} \in \mathrm{SU}(2)$. This is because $\operatorname{det} U_{k+1}^{(n)}=\operatorname{det} U_{k}^{(n)}=\operatorname{det} U_{0}^{(n)}=e^{i \varphi_{0}}$ since det $D_{j}= \pm 1$ and we have two instances of each $D_{j}$ matrix, meaning that all of the negative signs will cancel. Hence we can let $\varphi_{0}=0$, and by the argument in the previous section, if convergence holds for $\varphi_{0}=0$, then it holds for any $\varphi_{0}$. This also implies that for our selection of $\lambda_{j l}$, the sequence preserves the special unitary property of $U_{0}^{(n)}$.

We would like to show our elements come in the form $\left(U_{k+1}^{(n)}\right)_{11}=a_{k+1}^{(n)}=a_{k}^{(n)} \mathcal{A}_{k}^{(n)}$ and $\left(U_{k+1}^{(n)}\right)_{21}=b_{k+1}^{(n)}=$ $b_{k}^{(n)} \mathcal{B}_{k}^{(n)}$, where $\mathcal{A}_{k}^{(n)}$ and $\mathcal{B}_{k}^{(n)}$ are unknown functions in terms of $\lambda_{j l}$ and $b_{k}^{(n)}, a_{k}^{(n)}$ (later we will see that $b_{k}^{(n)}, a_{k}^{(n)}$ always come in the form $\left.\left|b_{k}^{(n)}\right|^{2},\left|a_{k}^{(n)}\right|^{2}\right)$. Our inductive sequence is created by taking $U_{k+1}^{(n)}$ and transforming it such that we map $D_{j} \mapsto D_{j+1}$ and $U_{k} \rightarrow U_{k}^{-1}$. This mapping is very easy to do since we take $b_{k} \mapsto-b_{k}, \overline{a_{k}} \mapsto a_{k}$, and $\lambda_{j l} \mapsto \lambda_{(j+1), l}$. The transformed sequence is denoted with a $T$ which is distributive across all terms and factors; for example, $T U_{k+1}^{(1)}=U_{k}^{-1} D_{2} U_{k} D_{2} U_{k}^{-1}$. The reason for doing so is that these can be easily inserted into the sequence $U_{k+1}^{(n+1)}$. Using our previous assumption on $a_{k+1}^{(n)}$ and $b_{k+1}^{(n)}$, this yields

$$
U_{k+1}^{(n)}=\left(\begin{array}{cc}
a_{k+1} & -\overline{b_{k+1}}  \tag{9}\\
b_{k+1} & \overline{a_{k+1}}
\end{array}\right) \equiv\left(\begin{array}{cc}
a_{k} \mathcal{A}_{k} & -\overline{b_{k} \mathcal{B}_{k}} \\
b_{k} \mathcal{B}_{k} & \overline{a_{k}} \overline{\mathcal{A}_{k}}
\end{array}\right) \Longrightarrow T U_{k+1}^{(n)}\left(\begin{array}{cc}
\overline{a_{k}} T \mathcal{A}_{k} & \overline{b_{k} T \mathcal{B}_{k}} \\
-b_{k} T \mathcal{B}_{k} & a_{k} \overline{T \mathcal{A}_{k}}
\end{array}\right)
$$

Now we insert this into our sequence $U_{k+1}^{(n+1)}$ :

$$
\begin{aligned}
& U_{k+1}^{(n+1)}=U_{k} \cdot D_{1} \cdot T U_{k+1}^{(n)} \cdot D_{1} \cdot U_{k}=\left(\begin{array}{cc}
a_{k} & -\overline{b_{k}} \\
b_{k} & \overline{a_{k}}
\end{array}\right)\left(\begin{array}{cc}
\lambda_{10} & 0 \\
0 & \lambda_{11}
\end{array}\right)\left(\begin{array}{cc}
\overline{a_{k} T \mathcal{A}_{k}} & \overline{b_{k} T \mathcal{B}_{k}} \\
-b_{k} T \mathcal{B}_{k} & a_{k} \overline{T \mathcal{A}_{k}}
\end{array}\right)\left(\begin{array}{cc}
\lambda_{10} & 0 \\
0 & \lambda_{11}
\end{array}\right)\left(\begin{array}{cc}
a_{k} & -\overline{b_{k}} \\
b_{k} & \overline{a_{k}}
\end{array}\right) \\
& =\left(\begin{array}{cc}
\lambda_{10} a_{k} & -\lambda_{11} \overline{b_{k}} \\
\lambda_{10} b_{k} & \lambda_{11} \overline{a_{k}}
\end{array}\right)\left(\begin{array}{cc}
\overline{a_{k}} T \mathcal{A}_{k} & \overline{b_{k} T \mathcal{B}_{k}} \\
-b_{k} T \mathcal{B}_{k} & a_{k} \overline{T \mathcal{A}_{k}}
\end{array}\right)\left(\begin{array}{cc}
\lambda_{10} a_{k} & -\lambda_{10} \overline{b_{k}} \\
\lambda_{11} b_{k} & \lambda_{11} \overline{a_{k}}
\end{array}\right) \\
& =\left(\begin{array}{cc}
\lambda_{10} a_{k} & -\lambda_{11} \overline{b_{k}} \\
\lambda_{10} b_{k} & \lambda_{11} \overline{a_{k}}
\end{array}\right)\left(\begin{array}{c}
\lambda_{10}\left|a_{k}\right|^{2} T \mathcal{A}_{k}+\lambda_{11}\left|b_{k}\right|^{2} \overline{T \mathcal{B}_{k}} \\
a_{k} b_{k}\left(\lambda_{11} \overline{T \mathcal{A}_{k}}-\lambda_{10} T \mathcal{B}_{k}\right) \\
\lambda_{11}\left|a_{k}\right|^{2} \overline{T \mathcal{A}_{k}}+\lambda_{10}\left|b_{k}\right|^{2} T \mathcal{B}_{k}
\end{array}\right) \\
& =\binom{\left(U_{k+1}^{(n+1)}\right)_{11}\left(U_{k+1}^{(n+1)}\right)_{12}}{\left(U_{k+1}^{(n+1)}\right)_{21}\left(U_{k+1}^{(n+1)}\right)_{22}} \\
& \left(U_{k+1}^{(n+1)}\right)_{11}=a_{k}\left(\lambda_{10}^{2}\left|a_{k}\right|^{2} T \mathcal{A}_{k}+\lambda_{10} \lambda_{11}\left|b_{k}\right|^{2} \overline{T \mathcal{B}_{k}}-\left|b_{k}\right|^{2}\left(\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}\right)\right) \\
& =a_{k}\left(\lambda_{10}^{2} T \mathcal{A}_{k}-\left|b_{k}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}+\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}\right)\right) \\
& \left(U_{k+1}^{(n+1)}\right)_{21}=b_{k}\left(\lambda_{10}^{2}\left|a_{k}\right|^{2} T \mathcal{A}_{k}+\lambda_{10} \lambda_{11}\left|b_{k}\right|^{2} \overline{T \mathcal{B}_{k}}+\left|a_{k}\right|^{2}\left(\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}\right)\right) \\
& =b_{k}\left(\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}+\left|a_{k}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}+\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}\right)\right) \\
& \left(U_{k+1}^{(n+1)}\right)_{12}=-\overline{b_{k}}\left(\lambda_{11}^{2}\left|a_{k}\right|^{2} \overline{T \mathcal{A}_{k}}+\lambda_{10} \lambda_{11}\left|b_{k}\right|^{2} T \mathcal{B}_{k}+\left|a_{k}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}\right)\right) \\
& =-\overline{b_{k}}\left(\lambda_{10} \lambda_{11} T \mathcal{B}_{k}+\left|a_{k}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}+\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}\right)\right) \\
& \left(U_{k+1}^{(n+1)}\right)_{22}=\overline{a_{k}}\left(\lambda_{11}^{2}\left|a_{k}\right|^{2} \overline{T \mathcal{A}_{k}}+\lambda_{10} \lambda_{11}\left|b_{k}\right|^{2} T \mathcal{B}_{k}-\left|b_{k}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}\right)\right) \\
& =\overline{a_{k}}\left(\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\left|b_{k}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}+\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}\right)\right) \\
& =\binom{a_{k}\left(\lambda_{10}^{2}\left|a_{k}\right|^{2} T \mathcal{A}_{k}+\lambda_{10} \lambda_{11}\left|b_{k}\right|^{2} \overline{T \mathcal{B}_{k}}-\left|b_{k}\right|^{2}\left(\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}\right)\right)-\overline{b_{k}}\left(\lambda_{11}^{2}\left|a_{k}\right|^{2} \overline{T \mathcal{A}_{k}}+\lambda_{10} \lambda_{11}\left|b_{k}\right|^{2} T \mathcal{B}_{k}+\left|a_{k}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}\right)\right)}{b_{k}\left(\lambda_{10}^{2}\left|a_{k}\right|^{2} T \mathcal{A}_{k}+\lambda_{10} \lambda_{11}\left|b_{k}\right|^{2} \overline{T \mathcal{B}_{k}}+\left|a_{k}\right|^{2}\left(\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}\right)\right) \overline{a_{k}}\left(\lambda_{11}^{2}\left|a_{k}\right|^{2} \overline{T \mathcal{A}_{k}}+\lambda_{10} \lambda_{11}\left|b_{k}\right|^{2} T \mathcal{B}_{k}-\left|b_{k}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}\right)\right)} \\
& =\left(\begin{array}{ccc}
a_{k}\left(\lambda_{10}^{2} T \mathcal{A}_{k}-\left|b_{k}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}+\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}\right)\right) & -\overline{k_{k}}\left(\lambda_{10} \lambda_{11} T \mathcal{B}_{k}+\left|a_{k}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}+\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}\right)\right) \\
b_{k}\left(\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}+\left|a_{k}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}+\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}\right)\right) & \overline{a_{k}}\left(\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\left|b_{k}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}+\lambda_{11}^{2} \overline{T \mathcal{A}_{k}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}}\right)\right)
\end{array}\right)
\end{aligned}
$$

Interestingly, if $\mathcal{A}_{k}^{(n)}$ and $\mathcal{B}_{k}^{(n)}$ are both functions of $\lambda_{j l}$ and $\left|b_{k}^{(n)}\right|^{2},\left|a_{k}^{(n)}\right|^{2}$, then so are $\mathcal{A}_{k}^{(n+1)}$ and $\mathcal{B}_{k}^{(n+1)}$ by examining the product above. Since this form is true for $n=1$, it must continue to hold for any $n$. Note that the matrix is still special unitary, and it preserves the form $\binom{a_{k}-\overline{b_{k}}}{b_{k}}$ that we initially gave on $U_{k}$.
3.1. Constructing a Recursive System of Equations. In order to simplify our equations a little more, we create the assumption that our sequences essentially construct polynomials in $\left|a_{k}\right|^{2},\left|b_{k}\right|^{2}$. We assume our
desired functions come in the slightly unusual form

$$
\begin{align*}
\mathcal{B}_{k}^{(n)} & =\beta_{0}^{(n)}+\left|a_{k}\right|^{2}\left(\beta_{1}^{(n)}-\left|b_{k}\right|^{2}\left(\beta_{2}^{(n)}+\left|a_{k}\right|^{2}\left(\beta_{3}^{(n)}-\left|b_{k}\right|^{2}(\ldots)\right)\right)\right) \\
& =\beta_{0}^{(n)}+\beta_{1}^{(n)}-\left|b_{k}\right|^{2}\left(\beta_{1}^{(n)}+\left|a_{k}\right|^{2}\left(\beta_{2}^{(n)}+\beta_{3}^{(n)}-\left|b_{k}\right|^{2}\left(\beta_{3}^{(n)}+\left|a_{k}\right|^{2}(\ldots)\right)\right)\right)  \tag{10}\\
\mathcal{A}_{k}^{(n)} & =\alpha_{0}^{(n)}-\left|b_{k}\right|^{2}\left(\alpha_{1}^{(n)}+\left|a_{k}\right|^{2}\left(\alpha_{2}^{(n)}-\left|b_{k}\right|^{2}\left(\alpha_{3}^{(n)}+\left|a_{k}\right|^{2}(\ldots)\right)\right)\right)  \tag{11}\\
& =\alpha_{0}^{(n)}-\alpha_{1}^{(n)}+\left|a_{k}\right|^{2}\left(\alpha_{1}^{(n)}-\left|b_{k}\right|^{2}\left(\alpha_{2}^{(n)}-\alpha_{3}^{(n)}+\left|a_{k}\right|^{2}\left(\alpha_{3}^{(n)}-\left|b_{k}\right|^{2}(\ldots)\right)\right)\right)
\end{align*}
$$

where the pattern stops at $\beta_{n}^{(n)}$ and $\alpha_{n}^{(n)}$. The second representation can be obtained from the first by applying the relation $\left|a_{k}\right|^{2}+\left|b_{k}\right|^{2}$ to each instance of $\left|a_{k}\right|^{2}$ and $\left|b_{k}\right|^{2}$. Here, $\alpha_{j}^{(n)}$ and $\beta_{j}^{(n)}$ are functions in $\lambda_{j l}$, defined differently for each $n$ and $j$. This form is preserved under induction by plugging in the forms and combining terms together. The second equality can easily be produced with induction and usage of the identity $\left|a_{k}\right|^{2}+\left|b_{k}\right|^{2}=1$.

There are two reasons we decided to assume this form: the first reason is that it allows for it to be easily inserted into the recursive form in the previous section, but also for the basic reason that, on observation, these forms seem to allow for the simplest forms of $\alpha_{j}^{(n)}$ and $\beta_{j}^{(n)}$. If we decided to go with some other polynomial in $\left|b_{k}\right|^{2}$ and $\left|a_{k}\right|^{2}$, the recursive equations would look different but ideally end up with the same desired properties. Our original sequence comes in the form

$$
\begin{align*}
& \frac{a_{k+1}^{(n+1)}}{a_{k}^{(n+1)}}=\mathcal{A}_{k}^{(n+1)}=\lambda_{10}^{2} T \mathcal{A}_{k}^{(n)}-\left|b_{k}^{(n+1)}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}^{(n)}+\lambda_{11}^{2} \overline{T \mathcal{A}_{k}^{(n)}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}^{(n)}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}^{(n)}}\right)  \tag{12}\\
& \frac{b_{k+1}^{(n+1)}}{b_{k}^{(n+1)}}=\mathcal{B}_{k}^{(n+1)}=\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}^{(n)}}+\left|a_{k}^{(n+1)}\right|^{2}\left(\lambda_{10}^{2} T \mathcal{A}_{k}^{(n)}+\lambda_{11}^{2} \overline{T \mathcal{A}_{k}^{(n)}}-\lambda_{10} \lambda_{11} T \mathcal{B}_{k}^{(n)}-\lambda_{10} \lambda_{11} \overline{T \mathcal{B}_{k}^{(n)}}\right) \tag{13}
\end{align*}
$$

Also note that we can assume $\mathcal{B}_{k}^{(n)}$ is real for any $k$ and $n$ by induction on $k$, since $\lambda_{10} \lambda_{11}=1$ and each complex term is summed with its conjugate, along with the fact that $\mathcal{B}_{0}^{(n)}=1$. Including this assumption allows us to create the following system of equations by aligning terms together and preserving the forms of $\mathcal{A}_{k}$ and $\mathcal{B}_{k}$ :

$$
\begin{gather*}
\alpha_{j}^{(n+1)}=\lambda_{10}^{2} T \alpha_{j}^{(n)}+\lambda_{10}^{2}\left(T \alpha_{j-1}^{(n)}-\chi_{j} T \alpha_{j}^{(n)}\right)+\lambda_{11}^{2}\left(\overline{T \alpha_{j-1}^{(n)}}-\chi_{j} \overline{T \alpha_{j}^{(n)}}\right)-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)}  \tag{14}\\
\beta_{j}^{(n+1)}=\lambda_{10} \lambda_{11} T \beta_{j}^{(n)}+\lambda_{10}^{2} T \alpha_{j-1}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11}\left(T \beta_{j-1}^{(n)}+\chi_{j} T \beta_{j}^{(n)}\right) \tag{15}
\end{gather*}
$$

We denote $\chi_{j} \equiv j \bmod 2$ as before, and so when we align terms from $\mathcal{A}_{j}^{(n)}, \mathcal{A}_{j-1}^{(n)}, \mathcal{B}_{j}^{(n)}$, and $\mathcal{B}_{j-1}^{(n)}$ we will have to use the second formulation for them, and this $\chi_{j}$ will alternate between having to add extra terms and not. Now we have reduced our complicated matrix into two equations, which are still very complicated, but in the next section we will see that under certain cases this simplifies a great deal. If $j$ is odd, then we get that this simplifies to

$$
\begin{align*}
& \alpha_{j}^{(n+1)}=\lambda_{10}^{2} T \alpha_{j-1}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)}-\lambda_{11}^{2} \overline{T \alpha_{j}^{(n)}} \\
& \beta_{j}^{(n+1)}=\lambda_{10}^{2} T \alpha_{j-1}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)}-\lambda_{10} \lambda_{11} T \beta_{j}^{(n)} \tag{16}
\end{align*}
$$

For even $j$, it becomes

$$
\begin{align*}
& \alpha_{j}^{(n+1)}=\lambda_{10}^{2} T \alpha_{j-1}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)}+\lambda_{10}^{2} T \alpha_{j}^{(n)}  \tag{17}\\
& \beta_{j}^{(n+1)}=\lambda_{10}^{2} T \alpha_{j-1}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)}+\lambda_{10} \lambda_{11} T \beta_{j}^{(n)}
\end{align*}
$$

Remark 3.1. It is worth noting that although our assumptions allow us to assume $U_{k}^{(n)} \in \mathrm{SU}(2)$ for any $k$, this is not necessary in order to find a convergent sequence. However, it is highly beneficial to do so since the recursive equations below are much easier to solve given our assumptions on $\lambda_{j l}$. Instead, we could provide some general form using functions $\mathcal{A}_{k}, \mathcal{B}_{k}, \mathcal{C}_{k}, \mathcal{D}_{k}$ :

$$
U_{k+1}^{(n)}=\left(\begin{array}{cc}
a_{k} \mathcal{A}_{k} & \overline{b_{k}} \mathcal{C}_{k} \\
b_{k} \mathcal{B}_{k} & \overline{a_{k}} \mathcal{D}_{k}
\end{array}\right)
$$

and we would have a set of 4 recursive equations to solve simultaneously. This might lead to more interesting general results for arbitrary $\lambda_{j l}$, but at the moment we are just concerned with proving Conjecture 2.1.

## 4. Deriving Necessary Values for Convergence

The most important part of these $\beta_{j}$ terms is that they equal the right values in the end, which turns out to be a complicated matter. To reiterate, we assume our end sequence $\mathcal{B}_{k}$ can be written as

$$
\mathcal{B}_{k}=\beta_{0}+\left|a_{k}\right|^{2}\left(\beta_{1}-\left|b_{k}\right|^{2}\left(\beta_{2}+\left|a_{k}\right|^{2}\left(\beta_{3}-\left|b_{k}\right|^{2}(\ldots)\right)\right)\right)
$$

This is quite an unusual form, so showing what values we want them to equal is a difficult matter. What would we like to happen? Using the identity $(1-x)\left(1+x+\ldots+x^{n-1}\right)=1-x^{n}$, we would like the terms to simplify to

$$
\begin{gathered}
\beta_{0}+\left|a_{k}\right|^{2}\left(\beta_{1}-\left|b_{k}\right|^{2}\left(\beta_{2}+\left|a_{k}\right|^{2}\left(\beta_{3}-\left|b_{k}\right|^{2}(\ldots)\right)\right)\right)=\beta_{0}+\left(1-\left|b_{k}\right|^{2}\right)\left(-\beta_{0}-\beta_{0}\left|b_{k}\right|^{2}-\beta_{0}\left|b_{k}\right|^{4}-\ldots-\beta_{0}\left|b_{k}\right|^{2 n-2}\right) \\
=\beta_{0}\left(1+\left(\left|b_{k}\right|^{2 n}-1\right)\right)=\beta_{0}\left|b_{k}\right|^{2 n}
\end{gathered}
$$

This would imply that $\left|b_{k+1}^{(n)}\right|=\left|b_{k}^{(n)} \mathcal{B}_{k}^{(n)}\right|=\left|\beta_{0}^{(n)}\right| \cdot\left|b_{k}^{(n)}\right|^{2 n+1}=\left|b_{k}^{(n)}\right|^{p}$ as desired, provided that $\left|\beta_{0}^{(n)}\right|=1$ (this is true as we will show below). As it turns out, all of the factors we wish for come in the form $\binom{n+k}{n}$, and these binomial coefficients have the nice relation that $\sum_{j=0}^{n}\binom{j+k}{j}=\binom{n+k+1}{n}$.
4.1. A Few Examples. It's reasonable to expect that we will get a linear system of equations from this, and here's the first few cases to demonstrate how this comes about.

$$
\begin{aligned}
\mathcal{B}_{k}^{(1)} & =\beta_{0}+\left|a_{k}\right|^{2}\left(\beta_{1}\right) \\
\mathcal{B}_{k}^{(2)} & =\beta_{0}+\left|a_{k}\right|^{2}\left(\beta_{1}-\left|b_{k}\right|^{2} \beta_{2}\right) \\
\mathcal{B}_{k}^{(4)} & =\beta_{0}+\left|a_{k}\right|^{2}\left(\beta_{1}-\left|b_{k}\right|^{2}\left(\beta_{2}+\left|a_{k}\right|^{2}\left(\beta_{3}-\left|b_{k}\right|^{2} \beta_{4}\right)\right)\right) \\
& =\beta_{0}+\left|a_{k}\right|^{2}\left(\beta_{1}+\beta_{2}\left(-\left|b_{k}\right|^{2}\right)+\beta_{3}\left(-\left|b_{k}\right|^{2}+\left|b_{k}\right|^{4}\right)+\beta_{4}\left(\left|b_{k}\right|^{4}-\left|b_{k}\right|^{6}\right)\right)
\end{aligned}
$$

To write this more suggestively, for $n=4$ the matrix system that solves this is, letting $v_{j} \equiv \beta_{j} / \beta_{0}$ :

$$
\left[\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{18}\\
0 & -1 & -1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & -1
\end{array}\right]\left[\begin{array}{l}
v_{1} \\
v_{2} \\
v_{3} \\
v_{4}
\end{array}\right]=\left[\begin{array}{l}
-1 \\
-1 \\
-1 \\
-1
\end{array}\right]
$$

Here, think of the $k$-th row corresponding to $\left|b_{k}\right|^{2 k-2}$. Inverting this matrix is possible since it has nonzero determinant, and in general this should always be possible for higher $n$, with the matrix coming in a nested form. Define each matrix that appears in this system as $M_{n}$, and so we have that $M_{4}$ is the matrix in the matrix system above. While this is perfectly reasonable to solve, it is quite complicated by hand. This complexity involving the binomial coefficients has to appear somewhere in the problem, and if we were to assume some other form on $\mathcal{B}_{k}$, the same result would still happen: the coefficients $v_{j}$ that are needed might be easier to find (say if it was a polynomial entirely in $\left|b_{k}\right|^{2}$, then we would just require all zeros except when $j=n$ ), but the function form is much more complicated to solve for. As we will see, even with the assumptions we give, the functions will still be enormously complicated.

Example 4.1. One interesting observation is that it appears $M_{m}$ contains $M_{n}$ in a nested sub-block fashion for $m>n$. Inverting the matrix $M_{n}$ is the main goal to solve the matrix system, and it helps to have an
example to see the patterns. Computationally inverting $M_{10}$ yields

$$
M_{10}=\left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{19}\\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -2 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -3 & -3 & -4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 6 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -4 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]
$$

and its inverse

$$
M_{10}^{-1}=\left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{20}\\
0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & 0 & 0 & 0 & -1 & -3 & -6 & -10 & -15 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 6 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -4 & -10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]
$$

4.2. Solving the Matrix System. A simple enough inductive argument can be given here. Essentially, we claim that when we add in $\beta_{n+1}$, we add in an additional column and row to the matrix that consists of entries from the Pascal triangle by converting $\left|a_{k}\right|^{2}$ to $1-\left|b_{k}\right|^{2}$.

Theorem 4.2. Denote $\vec{e} \in \mathbb{R}^{n}$ as the vector of all ones and $\vec{v}=\left(v_{1}, \ldots, v_{n}\right)$. Then the system $M_{n} \vec{v}=-\vec{e}$ is always solvable in integers for $\vec{v}$.

Proof. To begin, we also have to prove that $M_{n}$ can be defined in a natural way. For $M_{1}$ and $M_{2}$, these are immediately solvable as $\left(v_{1}^{(1)}\right)=(-1)$ and $\left(v_{1}^{(2)}, v_{2}^{(2)}\right)=(-1,1)$, and so this is the base case. We induct on $n$, but we will increment by 2 , proving two cases at a time.

Start by assuming that $n=2 q$ is even, and we already assume that $\mathcal{B}_{k}$ comes in the form above, which we can also express in a series as

$$
\begin{equation*}
\mathcal{B}_{k}^{(n)}=\beta_{0}^{(n)}+\left|a_{k}\right|^{2}\left(\sum_{j=1}^{q}\left(-\left|b_{k}\right|^{2}\left|a_{k}\right|^{2}\right)^{j-1}\left(\beta_{2 j-1}^{(n)}-\left|b_{k}\right|^{2} \beta_{2 j}^{(n)}\right)\right) \tag{21}
\end{equation*}
$$

We can expand $\mathcal{B}_{k}^{(n)}$ to $\mathcal{B}_{k}^{(n+1)}$ this by replacing $\beta_{2 q}$ with $\beta_{2 q}+\left|a_{k}\right|^{2} \beta_{2 q+1}$, and furthermore we expand $\mathcal{B}_{k}^{(n+1)}$ to $\mathcal{B}_{k}^{(n+2)}$ by replacing $\beta_{2 q+1}$ with $\beta_{2 q+1}-\left|b_{k}\right|^{2} \beta_{2 q+2}$ :

$$
\begin{aligned}
\mathcal{B}_{k}^{(n+2)} & =\beta_{0}^{(n+2)}+\left|a_{k}\right|^{2}\left(\sum_{j=1}^{q}\left(-\left|b_{k}\right|^{2}\left|a_{k}\right|^{2}\right)^{j-1}\left(\beta_{2 j-1}^{(n+2)}-\left|b_{k}\right|^{2} \beta_{2 j}^{(n+2)}\right)\right. \\
& \left.+\left(-\left|b_{k}\right|^{2}\left|a_{k}\right|^{2}\right)^{q-1}\left(-\left|b_{k}\right|^{2}\left|a_{k}\right|^{2}\right)\left(\beta_{2 q+1}^{(n+2)}-\left|b_{k}\right|^{2} \beta_{2 q+2}^{(n+2)}\right)\right) \\
& =\beta_{0}^{(n+2)}+\left|a_{k}\right|^{2}\left(\sum_{j=1}^{q+1}\left(-\left|b_{k}\right|^{2}\left|a_{k}\right|^{2}\right)^{j-1}\left(\beta_{2 j-1}^{(n+2)}-\left|b_{k}\right|^{2} \beta_{2 j}^{(n+2)}\right)\right)
\end{aligned}
$$

So our series formulation is valid. Using the binomial identity on $\left(-\left|a_{k}\right|^{2}\right)^{(j-1)}$ yields

$$
\left(-\left|a_{k}\right|^{2}\right)^{(j-1)}=\left(\left|b_{k}\right|^{2}-1\right)^{j-1}=\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}(-1)^{\ell}\left|b_{k}\right|^{2(j-1-\ell)}
$$

$$
\begin{equation*}
\sum_{j=1}^{q}\left(-\left|b_{k}\right|^{2}\left|a_{k}\right|^{2}\right)^{j-1}\left(\beta_{2 j-1}-\left|b_{k}\right|^{2} \beta_{2 j}\right)=\sum_{j=1}^{q} \sum_{\ell=0}^{j-1}\binom{j-1}{\ell}(-1)^{\ell}\left|b_{k}\right|^{2(j-1-\ell)}\left(\beta_{2 j-1}-\left|b_{k}\right|^{2} \beta_{2 j}\right) \tag{22}
\end{equation*}
$$

From this we can see that in general $\left[M_{n}\right]_{i i}$ is 1 for odd $i$ and -1 for even $i$ (this corresponds to $\ell=0$ in the summation above). This also confirms our matrix construction, which for $n=2 q$, increasing to $n=2(q+1)$ gives two more columns, the first of which has elements with entries $(-1)^{\ell}\binom{q}{\ell}$ and the second with entries $(-1)^{\ell+1}\binom{q}{\ell}$, where $\ell=0$ starts on the main diagonal, and increasing $\ell$ moves up above the diagonal by 1 . Since this is an upper diagonal matrix with nonzero entries on the diagonal, this is a matrix with nonzero determinant, so it is invertible. The inverse matrix is all integers due to the fact that the determinant is either $\pm 1$, and since the entries of the matrix are all integers, the inverse $M_{n}^{-1}=\operatorname{det}\left(M_{n}\right) C^{T}$ is also all integers since the determinant is an integer and the cofactor matrix $C$ consists of cofactors which are also all integers since $M_{n}$ is all integers. Multiplying both sides by the inverse gives $\vec{v}=-M_{n}^{-1} \vec{e}$, which is all integers.

We can write our matrix $M_{n}$ in the following manner by matching terms in the proof above: for $i \in \mathbb{N}$ and $j \geq 0$ we have $\left[M_{n}\right]_{i i}=(-1)^{i+1}$ and

$$
\begin{gather*}
{\left[M_{n}\right]_{(2 m+1-j),(2 m+1)}=(-1)^{j}\binom{m-1}{j}}  \tag{23}\\
{\left[M_{n}\right]_{(2 m-j),(2 m)}=(-1)^{j+1}\binom{m-1}{j}} \tag{24}
\end{gather*}
$$

Here we verify our guess for an inverse. The elements of the inverse entirely consist of the binomial coefficients $\binom{n+k}{n}$ we mentioned at the start:
Theorem 4.3. $M_{n}^{-1}$ has the following elements: $\left[M_{n}^{-1}\right]_{j j}=(-1)^{j+1}$ and the subsequent rows can be described as, for $i \in \mathbb{N}$ and $j \geq 0$ :

$$
\begin{gather*}
{\left[M_{n}^{-1}\right]_{(2 i),(2 i+j)}=-\binom{j+i-1}{j}}  \tag{25}\\
{\left[M_{n}^{-1}\right]_{(2 i+1),(2 i+1+j)}=\binom{j+i-1}{j}} \tag{26}
\end{gather*}
$$

The rest of the elements are 0 .
Proof. We can also rewrite the elements of the inverse as

$$
\begin{aligned}
{\left[M_{n}^{-1}\right]_{(2 i),(n-j)} } & =-\binom{n-i-j-1}{n-2 i-j} \\
{\left[M_{n}^{-1}\right]_{(2 i+1),(n-j)} } & =\binom{n-i-j-2}{n-2 i-j-1}
\end{aligned}
$$

We want to show that the product of the matrices is $X=M_{n}^{-1} M_{n}=I_{n}$. For an inductive argument, assume that the formula is true for $M_{n} . n=1$ and $n=2$ are the base cases, which have already been shown to be true. Then we can write $M_{n+1}$ as

$$
M_{n+1}=\left[\begin{array}{cc}
M_{n} & * \\
0 & (-1)^{n}
\end{array}\right]
$$

where $*$ is all the extra elements that would be included above the diagonal in the last column. The inverse is also an upper triangular matrix, and we know that the product of upper triangular matrices are upper triangular, so the inverse looks like

$$
M_{n+1}^{-1}=\left[\begin{array}{cc}
M_{n}^{-1} & * \\
0 & (-1)^{n}
\end{array}\right]
$$

And of course, the last element on the diagonal $X_{n+1, n+1}=1$ since the negative signs cancel. It remains to just show the last column above the diagonal is zeros. We split this into two cases, whether $n$ is odd or even. In either case, the first entry in the last column yields $X_{1, n+1}=0$, since we will never have a full column of nonzero elements in the last column of $M_{n}$ (the number increases by 1 every increment of $n$ by 2 ). The only nonzero element in the first row of the inverse is in the first column, but that will never be reached, so the product is 0 .
(1) If $n=2 m$, then the last column comes in the form

$$
\left[M_{n+1}\right]_{(2 m+1-j),(2 m+1)}=(-1)^{j}\binom{m-1}{j}
$$

and we want to multiply this column by every row in $M_{n+1}^{-1}$. For even rows $2 i$, this looks like

$$
X_{2 i, n+1}=-\sum_{j=0}^{n}(-1)^{j}\binom{m-1}{j}\binom{n-i-j}{n-2 i-j+1}=-\sum_{j=0}^{m-1}(-1)^{j}\binom{m-1}{j}\binom{2 m-i-j}{2 m-2 i-j+1}
$$

For odd rows $2 i+1$ it looks like

$$
X_{2 i+1, n+1}=\sum_{j=0}^{n}(-1)^{j}\binom{m-1}{j}\binom{n-i-j-1}{n-2 i-j}=\sum_{j=0}^{m-1}(-1)^{j}\binom{m-1}{j}\binom{2 m-i-j-1}{2 m-2 i-j}
$$

In both cases we are looking at $1 \leq i<m$.
(2) If $n=2 m-1$, then we get very similar situations:

$$
\left[M_{n+1}\right]_{(2 m-j),(2 m)}=(-1)^{j+1}\binom{m-1}{j}
$$

and we want to multiply this column by every row in $M_{n+1}^{-1}$. For even rows $2 i$, this looks like

$$
X_{2 i, n+1}=-\sum_{j=0}^{n}(-1)^{j+1}\binom{m-1}{j}\binom{n-i-j}{n-2 i-j+1}=\sum_{j=0}^{m-1}(-1)^{j}\binom{m-1}{j}\binom{2 m-i-j-1}{2 m-2 i-j}
$$

For odd rows $2 i+1$ it looks like
$X_{2 i+1, n+1}=\sum_{j=0}^{n}(-1)^{j+1}\binom{m-1}{j}\binom{n-i-j-1}{n-2 i-j}=-\sum_{j=0}^{m-1}(-1)^{j}\binom{m-1}{j}\binom{2 m-i-j-2}{2 m-2 i-j-1}$
For all of these it is worth noting that binomials of the form $\binom{\mu+\nu-1}{\mu-1}=\mu(\mu+1) \ldots(\mu+\nu-1) / \nu$ !, so for $0 \leq \nu<K$ this is a polynomial of degree less than or equal to $K$ in $\mu$. Let $\mu=2 m-2 i-j+R$, where $R=0,1,2$ (Each $R$ represents one of the three sums we have written), and $\nu=i-1$. In each case, $i-1<m-1$, so we have a polynomial with degree less than or equal to $m-1$. If $P(j)$ is a polynomial in $j$ with degree less than or equal to $d$, then

$$
\sum_{j=0}^{d}(-1)^{j}\binom{d}{j} P(j)=0
$$

This result is proven in [10], which can be applied to this very special case. Substitute $d \rightarrow m-1$, and $P(j) \rightarrow\binom{2 m-i-j+R-2}{2 m-2 i-j+R-1}$, which is a polynomial with degree less than or equal to $m-1$. This proves all the sums are 0 , and hence the last column is 0 above the diagonal whether $n$ is odd or even, and so by induction $X=I_{n}$ holds for all $n$. So $M_{n}^{-1}$ is the inverse of the matrix $M_{n}$.

From this we can solve the system of equations, given immediately as $\vec{v}=-M_{n}^{-1} \vec{e}$ :
Corollary 4.4. The solution to $M_{n} \vec{v}=-\vec{e}$ is given as

$$
\begin{equation*}
v_{1}=-1, \quad v_{2 i}=\sum_{j=0}^{n-2 i}\binom{j+i-1}{j}=\binom{n-i}{n-2 i}, \quad v_{2 i+1}=-\sum_{j=0}^{n-2 i-1}\binom{j+i-1}{j}=-\binom{n-i-1}{n-2 i-1} \tag{28}
\end{equation*}
$$

This yields the desired values these functions $\beta_{j}$ must take, as $\beta_{j}^{(n)}=\beta_{0}^{(n)} v_{j}^{(n)}$.

## 5. Using Recursive Equations to Solve Basic Sequences

Directly computing these formulas for a given $n$ by expanding $U_{k+1}^{(n)}$ is simple for smaller $p$, but in general the formula is not simple to derive. An inductive method works better for calculating high order cases as well as for analyzing. Here we compute some basic cases, and at the end we will show the explicit formulas $\beta_{j}^{(n)}$ and $\alpha_{j}^{(n)}$ for arbitrary $j$ and $n$.
Remark 5.1. It is worth noting that the results of the previous section and this section are two halves to the same problem. In order to completely prove the properties of our conjectured sequences, we must first derive what forms these $\beta_{j}^{(n)}$ and $\alpha_{j}^{(n)}$ functions take in terms of $\lambda_{j l}$, and second we show that, indeed, these functions equal our desired values when we simplify using our choices of $\lambda_{j l}$ in (6). This section is primarily concerned with the first half of the problem, not as much the second.

For $j=0$, the recursive equations (14) and (15) reduce to

$$
\begin{aligned}
& \alpha_{0}^{(n+1)}=\lambda_{10}^{2} T \alpha_{0}^{(n)} \\
& \beta_{0}^{(n+1)}=\lambda_{10} \lambda_{11} T \beta_{0}^{(n)}
\end{aligned}
$$

Theorem 5.2. $\alpha_{0}^{(n)}$ and $\beta_{0}^{(n)}$ are explicitly given by the formulas

$$
\begin{equation*}
\alpha_{0}^{(n)}=\prod_{k=1}^{n} \lambda_{k 0}^{2}=\omega^{\frac{1}{2}\left(2(-1)^{n} n+(-1)^{n}-1\right)}, \quad \beta_{0}^{(n)}=\prod_{k=1}^{n} \lambda_{k 0} \lambda_{k 1}=(-1)^{\frac{n(n+3)}{2}} \tag{29}
\end{equation*}
$$

Proof. For $p=3$ this corresponds to $n=1$, and explicitly calculating the sequence gives $\beta_{0}^{(1)}=\lambda_{10} \lambda_{11}$ and $\alpha_{0}^{(1)}=\lambda_{10}^{2}$, and so the above satisfies the initial conditions. Applying the translation operator $T$, we have

$$
\begin{gathered}
\lambda_{10}^{2} T \alpha_{0}^{(n)}=\lambda_{10}^{2} \prod_{k=2}^{n+1} \lambda_{k 0}^{2}=\prod_{k=1}^{n+1} \lambda_{k 0}^{2}=\alpha_{0}^{(n+1)} \\
\lambda_{10} \lambda_{11} T \beta_{0}^{(n)}=\lambda_{10} \lambda_{11} \prod_{k=2}^{n} \lambda_{k 0} \lambda_{k 1}=\prod_{k=1}^{n+1} \lambda_{k 0} \lambda_{k 1}=\beta_{0}^{(n+1)}
\end{gathered}
$$

$\lambda_{j 0}=\omega^{(-1)^{j} j}$ and $\lambda_{j 1}=(-1)^{j+1} \omega^{(-1)^{j+1} j}$, so we have that these reduce to the following when we plug in roots of unity. Let $(n-1)=2 q+r$ for $r=0,1$ :

$$
\begin{gathered}
\alpha_{0}^{(n)}=\prod_{k=1}^{n} \omega^{2(-1)^{k} k}=\omega^{2 \sum_{k=1}^{n}(-1)^{k} k}=\omega^{\frac{1}{2}\left(2(-1)^{n} n+(-1)^{n}-1\right)} \\
\beta_{0}^{(n)}=\prod_{k=1}^{n}(-1)^{k+1}=(-1)^{\sum_{k=1}^{n} k+1}=(-1)^{\frac{n(n+3)}{2}}=\sqrt{2} \cos \left(\frac{(2 n-1) \pi}{4}\right)
\end{gathered}
$$

These are strange sums, but they simplify quite nicely when you list out the first few iterations. For $n=1,2,3,4, \alpha_{0}^{(n)}=\omega^{-2}, \omega^{2}, \omega^{-4}, \omega^{4}$ and $\beta_{0}^{(n)}=1,-1,-1,1$, and the pattern continues in this convenient fashion. For $m \in \mathbb{N}$, then $\alpha_{0}^{(2 m)}=2 m, \alpha_{0}^{(2 m-1)}=-2 m, \beta_{0}^{(2 m)}=(-1)^{m}$, and $\beta_{0}^{(2 m-1)}=(-1)^{m+1}$.

The next is $j=n$, which is surprisingly easy to prove as well. For any $(n, k)$, we claim that $\alpha_{j}^{(n)}=\beta_{j}^{(n)}=0$ for $j>n$.

$$
\begin{aligned}
\alpha_{n+1}^{(n+1)} & =\lambda_{10}^{2} T \alpha_{n+1}^{(n)}+\lambda_{10}^{2}\left(T \alpha_{n}^{(n)}-\chi_{n+1} T \alpha_{n+1}^{(n)}\right) \\
& +\lambda_{11}^{2}\left(\overline{T \alpha_{n}^{(n)}}-\chi_{n+1} \overline{T \alpha_{n+1}^{(n)}}\right)-2 \lambda_{10} \lambda_{11} T \beta_{n}^{(n)} \\
& =\lambda_{10}^{2} T \alpha_{n}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{n}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{n}^{(n)} \\
\beta_{n+1}^{(n+1)} & =\lambda_{10} \lambda_{11} T \beta_{n+1}^{(n)}+\lambda_{10}^{2} T \alpha_{n}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{n}^{(n)}}-2 \lambda_{10} \lambda_{11}\left(T \beta_{n}^{(n)}+\chi_{n+1} T \beta_{n+1}^{(n)}\right) \\
& =\lambda_{10}^{2} T \alpha_{n}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{n}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{n}^{(n)}
\end{aligned}
$$

All of the cancelling above comes from the assumption that we gave before, that all functions are zero for $j>n$. We claim that both $\alpha_{n}^{(n)}$ and $\beta_{n}^{(n)}$ take the same form.

## Theorem 5.3.

$$
\begin{equation*}
\alpha_{n}^{(n)}=\beta_{n}^{(n)}=\prod_{k=1}^{n}\left(\lambda_{k 0}-\lambda_{k 1}\right)^{2} \tag{30}
\end{equation*}
$$

After plugging in roots of unity $\lambda_{j 1}=\omega^{(-1)^{j} j}$ and $\lambda_{j 1}=(-1)^{j+1} \omega^{(-1)^{j+1} j}, \beta_{n}^{(n)}$ simplifies to the expression

$$
\beta_{n} / \beta_{0}^{(n)}=(-1)^{\frac{n(n+3)}{2}} \prod_{k=1}^{n}\left(2 \cos (k \theta)+2(-1)^{k}\right)=(-1)^{n}
$$

Proof. Note that this is real, as we assume that $\lambda_{j 1}^{-1}=(-1)^{j} \lambda_{j 0}$ and $\lambda_{j 0}^{-1}=(-1)^{j} \lambda_{j 1}$, and since $D_{j}$ is diagonal, the eigenvalues are the elements of the diagonal, and we know the eigenvalues are always modulo 1 , so we have that taking the conjugate of $\left(\lambda_{j 0}-\lambda_{j 1}\right)^{2}$ yields the same function, so

$$
\lambda_{10}^{2} T \alpha_{n}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{n}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{n}^{(n)}=\left(\lambda_{10}^{2}+\lambda_{11}^{2}-2 \lambda_{10} \lambda_{11}\right) \prod_{k=2}^{n+1}\left(\lambda_{k 0}-\lambda_{k 1}\right)^{2}=\prod_{k=1}^{n+1}\left(\lambda_{k 0}-\lambda_{k 1}\right)^{2}
$$

So we have shown this for $j=n$. So we have the first two, but from there on it gets more difficult. Before moving on, we plug in our roots of unity:

$$
\begin{gathered}
\prod_{k=1}^{n}\left(\lambda_{k 0}-\lambda_{k 1}\right)^{2}=\prod_{k=1}^{n}\left(\omega^{(-1)^{k} k}+(-1)^{k} \omega^{(-1)^{k+1} k}\right)^{2}=\prod_{k=1}^{n}\left(\omega^{2(-1)^{k} k}+\omega^{2(-1)^{k+1} k}+2(-1)^{k}\right) \\
=\prod_{k=1}^{n}\left(2 \cos (k \theta)+2(-1)^{k}\right)=(-1)^{\frac{n(n+1)}{2}}
\end{gathered}
$$

The identity above is proved in the Appendix. Dividing by $\beta_{0}$ is the same as multiplying by it, and so we get that $v_{n}^{(n)}$ is equal to

$$
v_{n}^{(n)}=\beta_{n}^{(n)} / \beta_{0}^{(n)}=(-1)^{\frac{n(n+1)}{2}}(-1)^{\frac{n(n+3)}{2}}=(-1)^{n}
$$

Now we have $j=1$, in which case our inductive sequences are

$$
\begin{aligned}
& \alpha_{1}^{(n+1)}=\lambda_{10}^{2} T \alpha_{0}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{0}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{0}^{(n)}-\lambda_{11}^{2} \overline{T \alpha_{1}^{(n)}} \\
& \beta_{1}^{(n+1)}=\lambda_{10}^{2} T \alpha_{0}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{0}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{0}^{(n)}-\lambda_{10} \lambda_{11} T \beta_{1}^{(n)}
\end{aligned}
$$

However, we already know parts of this based on results above, so we can write

$$
\begin{aligned}
\alpha_{1}^{(n+1)} & =\prod_{k=1}^{n+1} \lambda_{k 0}^{2}+\prod_{k=1}^{n+1} \lambda_{k 1}^{2}-2 \prod_{k=1}^{n+1} \lambda_{k 0} \lambda_{k 1}-\lambda_{11}^{2} \overline{T \alpha_{1}^{(n)}} \\
& =\left(\prod_{k=1}^{n+1} \lambda_{k 0}-\prod_{k=1}^{n+1} \lambda_{k 1}\right)^{2}-\lambda_{11}^{2} \overline{T \alpha_{1}^{(n)}} \\
\beta_{1}^{(n+1)} & =\prod_{k=1}^{n+1} \lambda_{k 0}^{2}+\prod_{k=1}^{n+1} \lambda_{k 1}^{2}-2 \prod_{k=1}^{n+1} \lambda_{k 0} \lambda_{k 1}-\lambda_{10} \lambda_{11} T \beta_{1}^{(n)} \\
& =\left(\prod_{k=1}^{n+1} \lambda_{k 0}-\prod_{k=1}^{n+1} \lambda_{k 1}\right)^{2}-\lambda_{10} \lambda_{11} T \beta_{1}^{(n)}
\end{aligned}
$$

Theorem 5.4.

$$
\begin{align*}
& \alpha_{1}^{(n)}=\left(\prod_{k=1}^{n} \lambda_{k 0}-\prod_{k=1}^{n} \lambda_{k 1}\right)^{2}+\sum_{\ell=1}^{n-1}(-1)^{\ell} \prod_{k=1}^{\ell} \lambda_{k \chi_{k}}^{2}\left(\prod_{k=\ell+1}^{n} \lambda_{k 0}-\prod_{k=\ell+1}^{n} \lambda_{k 1}\right)^{2}  \tag{31}\\
& \beta_{1}^{(n)}=\left(\prod_{k=1}^{n} \lambda_{k 0}-\prod_{k=1}^{n} \lambda_{k 1}\right)^{2}+\sum_{\ell=1}^{n-1}(-1)^{\ell} \prod_{k=1}^{\ell} \lambda_{k 0} \lambda_{k 1}\left(\prod_{k=\ell+1}^{n} \lambda_{k 0}-\prod_{k=\ell+1}^{n} \lambda_{k 1}\right)^{2} \tag{32}
\end{align*}
$$

Proof. Since the top index of the sum is 0 for $n=1$, it doesn't really make sense, so just ignore the sum for this case, and so we have $\alpha_{1}^{(1)}=\left(\lambda_{10}-\lambda_{11}\right)^{2}$ and $\beta_{1}^{(1)}=\left(\lambda_{10}-\lambda_{11}\right)^{2}$ for the first case, which is what we already computed. Plugging into the recursion gives a very nice result:

$$
\begin{aligned}
& \lambda_{11}^{2} \overline{T \alpha_{1}^{(n)}}=\lambda_{11}^{2}\left(\prod_{k=2}^{n+1} \lambda_{k 0}-\prod_{k=2}^{n+1} \lambda_{k 1}\right)^{2}+\lambda_{11}^{2} \sum_{\ell=1}^{n-1}(-1)^{\ell} \prod_{k=2}^{\ell+1} \lambda_{k \chi_{k}}^{2}\left(\prod_{k=\ell+2}^{n+1} \lambda_{k 0}-\prod_{k=\ell+2}^{n+1} \lambda_{k 1}\right)^{2} \\
& =\lambda_{11}^{2}\left(\prod_{k=2}^{n+1} \lambda_{k 0}-\prod_{k=2}^{n+1} \lambda_{k 1}\right)^{2}+\sum_{\ell=1}^{n-1}(-1)^{\ell} \prod_{k=1}^{\ell+1} \lambda_{k \chi_{k}}^{2}\left(\prod_{k=\ell+2}^{n+1} \lambda_{k 0}-\prod_{k=\ell+2}^{n+1} \lambda_{k 1}\right)^{2} \\
& =\lambda_{11}^{2}\left(\prod_{k=2}^{n+1} \lambda_{k 0}-\prod_{k=2}^{n+1} \lambda_{k 1}\right)^{2}+\sum_{\ell=2}^{n}(-1)^{\ell-1} \prod_{k=1}^{\ell} \lambda_{k \chi_{k}}^{2}\left(\prod_{k=\ell+1}^{n+1} \lambda_{k 0}-\prod_{k=\ell+1}^{n+1} \lambda_{k 1}\right)^{2} \\
& =\sum_{\ell=1}^{n}(-1)^{\ell-1} \prod_{k=1}^{\ell} \lambda_{k \chi_{k}}^{2}\left(\prod_{k=\ell+1}^{n+1} \lambda_{k 0}-\prod_{k=\ell+1}^{n+1} \lambda_{k 1}\right)^{2} \\
& \lambda_{10} \lambda_{11} T \beta_{1}^{(n)}=\lambda_{10} \lambda_{11}\left(\prod_{k=2}^{n+1} \lambda_{k 0}-\prod_{k=2}^{n+1} \lambda_{k 1}\right)^{2}+\lambda_{10} \lambda_{11} \sum_{\ell=1}^{n-1}(-1)^{\ell} \prod_{k=2}^{\ell+1} \lambda_{k 0} \lambda_{k 1}\left(\prod_{k=\ell+2}^{n+1} \lambda_{k 0}-\prod_{k=\ell+2}^{n+1} \lambda_{k 1}\right)^{2} \\
& =\lambda_{10} \lambda_{11}\left(\prod_{k=2}^{n+1} \lambda_{k 0}-\prod_{k=2}^{n+1} \lambda_{k 1}\right)^{2}+\sum_{\ell=1}^{n-1}(-1)^{\ell} \prod_{k=1}^{\ell+1} \lambda_{k 0} \lambda_{k 1}\left(\prod_{k=\ell+2}^{n+1} \lambda_{k 0}-\prod_{k=\ell+2}^{n+1} \lambda_{k 1}\right)^{2} \\
& =\lambda_{10} \lambda_{11}\left(\prod_{k=2}^{n+1} \lambda_{k 0}-\prod_{k=2}^{n+1} \lambda_{k 1}\right)^{2}+\sum_{\ell=2}^{n}(-1)^{\ell-1} \prod_{k=1}^{\ell} \lambda_{k 0} \lambda_{k 1}\left(\prod_{k=\ell+1}^{n+1} \lambda_{k 0}-\prod_{k=\ell+1}^{n+1} \lambda_{k 1}\right)^{2} \\
& =\sum_{\ell=1}^{n}(-1)^{\ell-1} \prod_{k=1}^{\ell} \lambda_{k 0} \lambda_{k 1}\left(\prod_{k=\ell+1}^{n+1} \lambda_{k 0}-\prod_{k=\ell+1}^{n+1} \lambda_{k 1}\right)^{2}
\end{aligned}
$$

Of course, this means we get

$$
\begin{aligned}
\alpha_{1}^{(n+1)} & =\left(\prod_{k=1}^{n+1} \lambda_{k 0}-\prod_{k=1}^{n+1} \lambda_{k 1}\right)^{2}-\lambda_{11}^{2} \overline{T \alpha_{1}}(n) \\
& =\left(\prod_{k=1}^{n+1} \lambda_{k 0}-\prod_{k=1}^{n+1} \lambda_{k 1}\right)^{2}-\sum_{\ell=1}^{n}(-1)^{\ell-1} \prod_{k=1}^{\ell} \lambda_{k \chi_{k+1}}^{2}\left(\prod_{k=\ell+1}^{n+1} \lambda_{k 0}-\prod_{k=\ell+1}^{n+1} \lambda_{k 1}\right)^{2} \\
& =\left(\prod_{k=1}^{n+1} \lambda_{k 0}-\prod_{k=1}^{n+1} \lambda_{k 1}\right)^{2}+\sum_{\ell=1}^{n}(-1)^{\ell} \prod_{k=1}^{\ell} \lambda_{k \chi_{k+1}}^{2}\left(\prod_{k=\ell+1}^{n+1} \lambda_{k 0}-\prod_{k=\ell+1}^{n+1} \lambda_{k 1}\right)^{2} \\
\beta_{1}^{(n+1)} & =\left(\prod_{k=1}^{n+1} \lambda_{k 0}-\prod_{k=1}^{n+1} \lambda_{k 1}\right)^{2}-\lambda_{10} \lambda_{11} T \beta_{1}^{(n)} \\
& =\left(\prod_{k=1}^{n+1} \lambda_{k 0}-\prod_{k=1}^{n+1} \lambda_{k 1}\right)^{2}-\sum_{\ell=1}^{n}(-1)^{\ell-1} \prod_{k=1}^{\ell} \lambda_{k 0} \lambda_{k 1}\left(\prod_{k=\ell+1}^{n+1} \lambda_{k 0}-\prod_{k=\ell+1}^{n+1} \lambda_{k 1}\right)^{2} \\
& =\left(\prod_{k=1}^{n+1} \lambda_{k 0}-\prod_{k=1}^{n+1} \lambda_{k 1}\right)^{2}+\sum_{\ell=1}^{n}(-1)^{\ell} \prod_{k=1}^{n} \lambda_{k 0} \lambda_{k 1}\left(\prod_{k=\ell+1}^{n+1} \lambda_{k 0}-\prod_{k=\ell+1}^{n+1} \lambda_{k 1}\right)^{2}
\end{aligned}
$$

And this confirms our formula by induction.

Remark 5.5. In abuse of notation, introduce variables $\lambda_{00}=\lambda_{01}=1$, and so we can write the formulas as

$$
\begin{align*}
& \alpha_{1}^{(n)}=\sum_{\ell=0}^{n-1}(-1)^{\ell} \prod_{k=0}^{\ell} \lambda_{k \chi_{k}}^{2}\left(\prod_{k=\ell+1}^{n} \lambda_{k 0}-\prod_{k=\ell+1}^{n} \lambda_{k 1}\right)^{2}  \tag{33}\\
& \beta_{1}^{(n)}=\sum_{\ell=0}^{n-1}(-1)^{\ell} \prod_{k=0}^{\ell} \lambda_{k 0} \lambda_{k 1}\left(\prod_{k=\ell+1}^{n} \lambda_{k 0}-\prod_{k=\ell+1}^{n} \lambda_{k 1}\right)^{2} \tag{34}
\end{align*}
$$

Applying the shifting operation doesn't really make sense here since $\lambda_{00}, \lambda_{01}$ aren't really variables the shifting operator works on. However, if we multiply by $\lambda_{10} \lambda_{11}$ or what respective factor is in front of the shift, all is well since this takes the place of the $\lambda_{00}, \lambda_{01}$, which are now gone:

$$
\begin{aligned}
\lambda_{11}^{2} \overline{T \alpha_{1}^{(n)}}= & \sum_{\ell=1}^{n}(-1)^{\ell-1} \prod_{k=1}^{\ell} \lambda_{k \chi_{k}}^{2}\left(\prod_{k=\ell+1}^{n+1} \lambda_{k 0}-\prod_{k=\ell+1}^{n+1} \lambda_{k 1}\right)^{2} \\
\lambda_{10} \lambda_{11} T \beta_{1}^{(n)}= & \sum_{\ell=1}^{n}(-1)^{\ell-1} \prod_{k=1}^{\ell} \lambda_{k 0} \lambda_{k 1}\left(\prod_{k=\ell+1}^{n+1} \lambda_{k 0}-\prod_{k=\ell+1}^{n+1} \lambda_{k 1}\right)^{2}
\end{aligned}
$$

This may not seem necessary, but it makes the expression shorter to write, which we shall see is very helpful when writing down these formulas, which will get more and more complex.

Plugging in for $\lambda_{j 0}$ and $\lambda_{j 1}$ gives the expression

$$
\begin{gathered}
\beta_{1}^{(n)}=\left(\prod_{k=1}^{n} \omega^{2(-1)^{k} k}+\prod_{k=1}^{n} \omega^{2(-1)^{k+1} k}-2(-1)^{\frac{n(n+3)}{2}}\right) \\
+\sum_{\ell=1}^{n-1}(-1)^{\frac{\ell(\ell+1)}{2}}\left(\prod_{k=\ell+1}^{n} \omega^{2(-1)^{k} k}+\prod_{k=\ell+1}^{n} \omega^{2(-1)^{k+1} k}-2(-1)^{\frac{n(n+3)}{2}-\frac{\ell(\ell+3)}{2}}\right)
\end{gathered}
$$

This doesn't seem simple, but up to some rearranging the formulas are quite nice.
Theorem 5.6. After plugging in roots of unity $\lambda_{j 0}=\omega^{(-1)^{j} j}$ and $\lambda_{j 1}=(-1)^{j+1} \omega^{(-1)^{j+1} j}$, $\beta_{1}$ always equals the correct value -1 when factoring out $\beta_{0}$ :

$$
\begin{equation*}
\beta_{1}^{(2 m)} / \beta_{0}^{(2 m)}=2 \sum_{k=1}^{2 m}(-1)^{k} \cos (k \theta)=-1, \quad \beta_{1}^{(2 m-1)} / \beta_{0}^{(2 m-1)}=2 \sum_{k=0}^{2 m-1}(-1)^{k+1} \cos (k \theta)=-1 \tag{35}
\end{equation*}
$$

Proof. We can consider this in two cases: $n=2 m$ and $2 m-1$. For $n=2 m$, combining all under one sum gives $(2 m(2 m+3) / 2 \cong m$ and $\ell(\ell+1) / 2-\ell(\ell+3) / 2 \cong \ell$ as powers of -1$)$

$$
\beta_{1}^{(2 m)}=\sum_{\ell=0}^{2 m-1}(-1)^{\frac{\ell(\ell+1)}{2}}\left(\prod_{k=\ell+1}^{2 m} \omega^{2(-1)^{k} k}+\prod_{k=\ell+1}^{2 m} \omega^{2(-1)^{k+1} k}\right)-2(-1)^{m+\ell}
$$

For the last term, every two terms together are equal to $2(-1)^{m}+2(-1)^{m+1}$, and so adding all of them up is equal to 0 since we are summing over an even number of terms. As for the first two terms, we can rearrange these. For $\ell=0$, we know that

$$
\prod_{k=1}^{2 m} \omega^{2(-1)^{k} k}+\prod_{k=1}^{2 m} \omega^{2(-1)^{k+1} k}=\omega^{2 m}+\omega^{-2 m}=2 \cos (m \theta)
$$

Say that $\ell=2 l$, then using the same product as above, we get

$$
\prod_{k=2 l+1}^{2 m} \omega^{2(-1)^{k} k}+\prod_{k=1}^{2 m} \omega^{2(-1)^{k+1} k}=\omega^{2(m-l)}+\omega^{-2(m-l)}=2 \cos ((m-l) \theta)
$$

And of course using the formulas from $j=0$, if $\ell=2 l-1$, then the product is equal to $2 \cos ((m+l) \theta)$. In this fashion, we iterate through every single power 1 to $n$ in the sequence $m, m+1, m-1, m+2, m-2, \ldots$, and so we will have terms $\cos (k \theta)$, where $k=1, \ldots, n$. What about the signs in front? Plugging in $\ell=0$,
$(-1)^{\ell(\ell+1) / 2}=1$, and so the $\operatorname{sign}$ for $\cos (m \theta)$ is always positive. If we let $\ell=2 l$, then we have the sign as $(-1)^{l}$, and if $\ell=2 l-1$, then the sign is also $(-1)^{l}$, and so the sign alternates from $k=1$ to $n$ in front of $\cos (k \theta)$, but we know that the sign is always positive for $m$. So whatever parity $m$ is, when $k$ has the same parity the sign is the same. We write this as

$$
\beta_{1}^{(2 m)}=2(-1)^{m} \sum_{k=1}^{2 m}(-1)^{k} \cos (k \theta)
$$

The sign alternates, and we always have $\mathrm{a}+\cos (m \theta)$ term by including the $(-1)^{m}$ factor. However, $\beta_{0}^{(2 m)}=$ $(-1)^{m}$, and so by Lemma A. 1

$$
v_{1}^{(2 m)}=\beta_{1}^{(2 m)} / \beta_{0}^{(2 m)}=2 \sum_{k=1}^{2 m}(-1)^{k} \cos (k \theta)=-1
$$

For $n=2 m-1$, we get a very similar equation. $n(n+3) / 2 \cong m+1$, so

$$
\beta_{1}^{(2 m-1)}=\sum_{\ell=0}^{2 m-2}(-1)^{\frac{\ell(\ell+1)}{2}}\left(\prod_{k=\ell+1}^{2 m-1} \omega^{2(-1)^{k} k}+\prod_{k=\ell+1}^{2 m-1} \omega^{2(-1)^{k+1} k}\right)-2(-1)^{m+\ell+1}
$$

Similar to before, the alternating sign cancels with itself except for $\ell=0$, which gives $-2(-1)^{m+1}$. For $\ell=0$, we have that similar to before, we always have a term $+\cos (m \theta)$, since the total product from 1 to $2 m-1$ gives $-2 m$, and combining with the conjugate gives $\cos (m \theta)$. The sign is positive for the same reason as before. This time, we have that increasing by $\ell=2 l$ increases our frequency, $\cos ((m+l) \theta)$. This is because we decrease by 1 , but since the initial $\ell=0$ was negative, we are increasing the magnitude of the frequency. In the same way, we decrease the frequency with $\ell=2 l-1$ giving $\cos ((m-l) \theta)$. In this way, we cover every $\cos (k \theta)$ from $k=1, \ldots, n$, and the signs alternate in the exact way as above. This time, $\cos (m \theta)$ is always positive, and so combining everything we have

$$
\begin{gathered}
\beta_{1}^{(2 m-1)}=-2(-1)^{m+1}+2(-1)^{m} \sum_{k=1}^{2 m-1}(-1)^{k} \cos (k \theta)=(-1)^{m+1}\left(-2-2 \sum_{k=1}^{2 m-1}(-1)^{k} \cos (k \theta)\right) \\
=2(-1)^{m+1} \sum_{k=0}^{2 m-1}(-1)^{k+1} \cos (k \theta)
\end{gathered}
$$

Factoring out $\beta_{0}^{(2 m-1)}=(-1)^{m+1}$ yields

$$
\beta_{1} / \beta_{0}^{(2 m-1)}=2 \sum_{k=0}^{2 m-1}(-1)^{k+1} \cos (k \theta)=-1
$$

This completes the proof, as we have now shown $\forall n, \beta_{1}^{(n)} / \beta_{0}^{(n)}=-1$.
The cases $j=2$ and beyond are much more complicated, and so we will leave them for a different paper. What we can do here, though, is derive the formulas themselves in terms of $\lambda_{j l}$.
5.1. Arbitrary Order. To summarize everything that we have, we will have an enormously long expression to handle. In general, we should expect the higher cases to be handled with one term as a nested sum. There should likely be $j$ summation signs and a summand which is a product of $j$ factors. To restate our recursive equations (14) and (15),

$$
\begin{gathered}
\alpha_{j}^{(n+1)}=\lambda_{10}^{2} T \alpha_{j}^{(n)}+\lambda_{10}^{2}\left(T \alpha_{j-1}^{(n)}-\chi_{j} T \alpha_{j}^{(n)}\right)+\lambda_{11}^{2}\left(\overline{T \alpha_{j-1}^{(n)}}-\chi_{j} \overline{T \alpha_{j}^{(n)}}\right)-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)} \\
\beta_{j}^{(n+1)}=\lambda_{10} \lambda_{11} T \beta_{j}^{(n)}+\lambda_{10}^{2} T \alpha_{j-1}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11}\left(T \beta_{j-1}^{(n)}+\chi_{j} T \beta_{j}^{(n)}\right)
\end{gathered}
$$

We give the following result, which describes the complete solution to these equations.

Theorem 5.7. The general solution for $\alpha_{j}^{(n)}$ and $\beta_{j}^{(n)}$ for $j \leq n$ is given by the following. Denote $L_{j_{1}}^{j_{2}}=$ $\sum_{j=j_{1}}^{j_{2}} \ell_{j}$, where $\ell_{j}$ are indices of summation. Then for $j \geq 1$,

$$
\begin{align*}
& \alpha_{j}^{(n)}=\sum_{\ell_{j}=0}^{n-j}\left[\prod_{k=0}^{\ell_{j}} \lambda_{k, \chi_{j} \chi_{k}}^{2}\right] \sum_{\ell_{j-1}=0}^{n-j-L_{j}^{j}} \sum_{\ell_{j-2}=0}^{n-j-L_{j-1}^{j}} \cdots \sum_{\ell_{1}=0}^{n-j-L_{2}^{j}}\left(\prod_{k=L_{1}^{j}+j}^{n} \lambda_{k 0}-\prod_{k=L_{1}^{j}+j}^{n} \lambda_{k 1}\right)^{2} \\
& \times\left[\prod_{\mu=1}^{j}(-1)^{\chi_{\mu} \ell_{\mu}}\right] \times\left[\prod_{\mu=1}^{j-1}\left(\prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(\chi_{\mu} \chi_{k}\right)} \prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j+j-\mu}} \lambda_{k,\left(1-\chi_{\mu} \chi_{k}\right)}\right)^{2}\right]  \tag{36}\\
& \beta_{j}^{(n)}=\sum_{\ell_{j}=0}^{n-j}\left[\prod_{k=0}^{\ell_{j}} \lambda_{k 0} \lambda_{k 1}\right] \sum_{\ell_{j-1}=0}^{n-j-L_{j}^{j}} \sum_{\ell_{j-2}=0}^{n-j-L_{j-1}^{j}} \cdots \sum_{\ell_{1}=0}^{n-j-L_{2}^{j}}\left(\prod_{k=L_{1}^{j}+j}^{n} \lambda_{k 0}-\prod_{k=L_{1}^{j}+j}^{n} \lambda_{k 1}\right)^{2} \\
& \times\left[\prod_{\mu=1}^{j}(-1)^{\chi_{\mu} \ell_{\mu}}\right] \times\left[\prod_{\mu=1}^{j-1}\left(\prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(\chi_{\mu} \chi_{k}\right)} \prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(1-\chi_{\mu} \chi_{k}\right)}\right)^{2}\right] \tag{37}
\end{align*}
$$

Proof. The proof is by double induction, using the base cases $(j, n)=(1, n)$ and $(n, n)$, and we will use the recursive equations to prove the formulas are true for $(j, n+1)$, assuming $(j-1, n)$ and $(j, n)$. Plugging in $j=1$ leaves only the first sum, with $\ell_{2}=0$. The second product can be ignored, and $L_{1}^{j}+j=\ell_{1}+1$. This reduces exactly to the correct formulas for $j=1$ given in (33) and (34), similarly so for $j=n$.
(1) If $j=2 i$, then the equation we are considering is

$$
\begin{aligned}
\alpha_{j}^{(n+1)} & =\lambda_{10}^{2} T \alpha_{j-1}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)}+\lambda_{10}^{2} T \alpha_{j}^{(n)} \\
\beta_{j}^{(n+1)} & =\lambda_{10}^{2} T \alpha_{j-1}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)}+\lambda_{10} \lambda_{11} T \beta_{j}^{(n)}
\end{aligned}
$$

It suffices to just check $\alpha_{j}$ since the difference between the two formulas is just the factors in front. The first three terms can be written as follows since $j-1$ is odd.

$$
\begin{gathered}
\lambda_{10}^{2} T \alpha_{j-1}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)} \\
=\sum_{\ell_{j-1}=0}^{n-j+1}\left(\prod_{k=1}^{\ell_{j-1}+1} \lambda_{k \chi_{k+1}}-\prod_{k=1}^{\ell_{j-1}+1} \lambda_{k \chi_{k}}\right)_{\ell_{j-2}=0}^{n-j+1-L_{j-1}^{j-1}} \cdots \sum_{\ell_{1}=0}^{n-j+1-L_{2}^{j-1}}\left(\prod_{k=L_{1}^{j-1}+j}^{n+1} \lambda_{k 0}-\prod_{k=L_{1}^{j-1}+j}^{n+1} \lambda_{k 1}\right)^{2} \\
\times\left[\prod_{\mu=1}^{j-1}(-1)^{\chi_{\mu} \ell_{\mu}}\right] \times\left[\prod_{\mu=1}^{j-2}\left(\prod_{k=L_{1+\mu}^{j-1}+j-\mu}^{L_{\mu}^{j-1}+j-\mu} \lambda_{k,\left(\chi_{\mu} \chi_{k}\right)}-\prod_{k=L_{1+\mu}^{j-1}+j-\mu}^{L_{\mu}^{j-1}+j-\mu} \lambda_{k,\left(1-\chi_{\mu} \chi_{k}\right)}\right)^{2}\right]
\end{gathered}
$$

Each of the product indices are raised by 1 due to $T$, and the terms in the last factors of the expression are swapped from this operation, although the expression is the same. Now we can place the factor in the front into the product at the end, where it takes the place of $\mu=j-1$. We also raise the powers of negative signs by 1 since $j$ is even, and so we are just multiplying by 1 . Let $\ell_{j}=0$ so that we can write the index $\operatorname{sum} L_{*}^{j-1}$ as $L_{*}^{j}$ :

$$
\begin{gathered}
=\sum_{\ell_{j-1}=0}^{n-j+1} \sum_{\ell_{j-2}=0}^{n-j+1-L_{j-1}^{j}} \cdots \sum_{\ell_{1}=0}^{n-j+1-L_{2}^{j}}\left(\prod_{k=L_{1}^{j}+j}^{n+1} \lambda_{k 0}-\prod_{k=L_{1}^{j}+j}^{n+1} \lambda_{k 1}\right)^{2} \\
\times\left[\prod_{\mu=1}^{j}(-1)^{\chi_{\mu} \ell_{\mu}}\right] \times\left[\prod_{\mu=1}^{j-1}\left(\prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(\chi_{\mu} \chi_{k}\right)}^{n} \prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j+j-\mu}} \lambda_{k,\left(1-\chi_{\mu} \chi_{k}\right)}\right)^{2}\right]
\end{gathered}
$$

The last term is

$$
\begin{aligned}
& \lambda_{10}^{2} T \alpha_{j}^{(n)}=\sum_{\ell_{j}=0}^{n-j}\left[\prod_{k=0}^{\ell_{j}+1} \lambda_{k 0}^{2}\right] \sum_{\ell_{j-1}=0}^{n-j-L_{j}^{j}} \sum_{\ell_{j-2}=0}^{n-j-L_{j-1}^{j}} \cdots \sum_{\ell_{1}=0}^{n-j-L_{2}^{j}}\left(\prod_{k=L_{1}^{j}+j+1}^{n+1} \lambda_{k 0}-\prod_{k=L_{1}^{j}+j+1}^{n+1} \lambda_{k 1}\right)^{2} \\
& \times\left[\prod_{\mu=1}^{j}(-1)^{\chi_{\mu} \ell_{\mu}}\right] \times\left[\prod_{\mu=1}^{j-1}\left(\prod_{k=L_{1+\mu}^{j}+j+1-\mu}^{L_{\mu}^{j}+j+1-\mu} \lambda_{k,\left(\chi_{\mu} \chi_{k}\right)}^{L_{\mu}^{j}+j+1-\mu} \prod_{k=L_{1+\mu}^{j}+j+1-\mu}^{L_{\mu}^{j}} \lambda_{k,\left(1-\chi_{\mu} \chi_{k}\right)}\right)^{2}\right] \\
& =\sum_{\ell_{j}=1}^{n-j+1}\left[\prod_{k=0}^{\ell_{j}} \lambda_{k 0}^{2}\right] \sum_{\ell_{j-1}=0}^{n-j+1-L_{j}^{j}} \sum_{\ell_{j-2}=0}^{n-j+1-L_{j-1}^{j}} \cdots \sum_{\ell_{1}=0}^{n-j+1-L_{2}^{j}}\left(\prod_{k=L_{1}^{j}+j}^{n+1} \lambda_{k 0}-\prod_{k=L_{1}^{j}+j}^{n+1} \lambda_{k 1}\right)^{n} \\
& \times\left[\prod_{\mu=1}^{j}(-1)^{\chi_{\mu} \ell_{\mu}}\right] \times\left[\prod_{\mu=1}^{j-1}\left(\prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(\chi_{\mu} \chi_{k}\right)} \prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(1-\chi_{\mu} \chi_{k}\right)}\right)^{2}\right]
\end{aligned}
$$

Adding the terms together yields our desired result by letting $\ell_{j}=0$ in the formula for $\lambda_{10}^{2} T \alpha_{j-1}^{(n)}+$ $\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)}$. These terms take the $\ell_{j}$ place in the total sum.

$$
\begin{aligned}
& \alpha_{j}^{(n+1)}=\sum_{\ell_{j}=0}^{n-j+1}\left[\prod_{k=0}^{\ell_{j}} \lambda_{k 0}^{2}\right] \sum_{\ell_{j-1}=0}^{n-j+1-L_{j}^{j}} \sum_{\ell_{j-2}=0}^{n-j+1-L_{j-1}^{j}} \cdots \sum_{\ell_{1}=0}^{n-j+1-L_{2}^{j}}\left(\prod_{k=L_{1}^{j}+j}^{n+1} \lambda_{k 0}-\prod_{k=L_{1}^{j}+j}^{n+1} \lambda_{k 1}\right)^{2} \\
& \times\left[\prod_{\mu=1}^{j}(-1)^{\chi_{\mu} \ell_{\mu}}\right] \times\left[\prod_{\mu=1}^{j-1}\left(\prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(\chi_{\mu} \chi_{k}\right)} \prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j+j-\mu}} \lambda_{k,\left(1-\chi_{\mu} \chi_{k}\right)}\right)^{2}\right]
\end{aligned}
$$

(2) If $j=2 i+1$, then the equation we are considering is

$$
\begin{aligned}
\alpha_{j}^{(n+1)} & =\lambda_{10}^{2} T \alpha_{j-1}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)}-\lambda_{11}^{2} \overline{T \alpha_{j}^{(n)}} \\
\beta_{j}^{(n+1)} & =\lambda_{10}^{2} T \alpha_{j-1}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)}-\lambda_{10} \lambda_{11} T \beta_{j}^{(n)}
\end{aligned}
$$

It suffices to check $\alpha_{j}$, for the same reasons as above. In this case $j-1$ is even, so we get

$$
\begin{aligned}
& \lambda_{10}^{2} T \alpha_{j-1}^{(n)}+\lambda_{11}^{2} \overline{T \alpha_{j-1}^{(n)}}-2 \lambda_{10} \lambda_{11} T \beta_{j-1}^{(n)} \\
& =\sum_{\ell_{j-1}=0}^{n-j+1}\left(\prod_{k=1}^{\ell_{j-1}+1} \lambda_{k 0}-\prod_{k=1}^{\ell_{j-1}+1} \lambda_{k 1}\right)^{2} \sum_{\ell_{j-2}=0}^{n-j+1-L_{j-1}^{j-1}} \cdots \sum_{\ell_{1}=0}^{n-j+1-L_{2}^{j-1}}\left(\prod_{k=L_{1}^{j-1}+j}^{n+1} \lambda_{k 0}-\prod_{k=L_{1}^{j-1}+j}^{n+1} \lambda_{k 1}\right)^{2} \\
& \times\left[\prod_{\mu=1}^{j-1}(-1)^{\chi_{\mu} \ell_{\mu}}\right] \times\left[\prod_{\mu=1}^{j-2}\left(\prod_{k=L_{1+\mu}^{j-1}+j-\mu}^{L_{\mu}^{j-1}+j-\mu} \lambda_{k,\left(\chi_{\mu} \chi_{k}\right)}-\prod_{k=L_{1+\mu}^{j-1}+j-\mu}^{L_{\mu}^{j-1}+j-\mu} \lambda_{k,\left(1-\chi_{\mu} \chi_{k}\right)}\right)^{2}\right] \\
& =\sum_{\ell_{j-1}=0}^{n-j+1} \sum_{\ell_{j-2}=0}^{n-j+1-L_{j-1}^{j}} \cdots \sum_{\ell_{1}=0}^{n-j+1-L_{2}^{j}}\left(\prod_{k=L_{1}^{j}+j}^{n+1} \lambda_{k 0}-\prod_{k=L_{1}^{j}+j}^{n+1} \lambda_{k 1}\right)^{2} \\
& \times\left[\prod_{\mu=1}^{j}(-1)^{\chi_{\mu} \ell_{\mu}}\right] \times\left[\prod_{\mu=1}^{j-1}\left(\prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(\chi_{\mu} \chi_{k}\right)}-\prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(1-\chi_{\mu} \chi_{k}\right)}\right)^{2}\right]
\end{aligned}
$$

As before, we have let $\ell_{j}=0$ above to make the expression easier to combine with the last term, which can be written as

$$
\begin{aligned}
& -\lambda_{11}^{2} \overline{T \alpha_{j}^{(n)}}=-\sum_{\ell_{j}=0}^{n-j}\left[\prod_{k=0}^{\ell_{j}+1} \lambda_{k \chi_{k}}^{2}\right] \sum_{\ell_{j-1}=0} \sum_{\ell_{j-2}=0}^{n-j-L_{j}^{j}} \cdots \sum_{\ell_{1}=0}^{n-j-L_{j-1}^{j}}\left(\prod_{k=L_{1}^{j}+j+1}^{n-j-L_{2}^{j}} \lambda_{k 0}^{n+1} \prod_{k=L_{1}^{j}+j+1}^{n+1} \lambda_{k 1}\right)^{2} \\
& \times(-1)^{\ell_{j}}\left[\prod_{\mu=1}^{j-1}(-1)^{\chi_{\mu} \ell_{\mu}}\right] \times\left[\prod_{\mu=1}^{j-1}\left(\prod_{k=L_{1+\mu}^{j}+j+1-\mu}^{L_{\mu}^{j}+j+1-\mu} \lambda_{k,\left(\chi_{\mu} \chi_{k}\right)} \prod_{k=L_{1+\mu}^{j}+j+1-\mu}^{L_{\mu}^{j}+j+1-\mu} \lambda_{k,\left(1-\chi_{\mu} \chi_{k}\right)} \prod^{2}\right]\right. \\
& =\sum_{\ell_{j}=1}^{n-j+1}\left[\prod_{k=0}^{\ell_{j}} \lambda_{k \chi_{k}}^{2}\right] \sum_{\ell_{j-1}=0}^{n-j+1-L_{j}^{j}} \sum_{\ell_{j-2}=0}^{n-j+1-L_{j-1}^{j}} \cdots \sum_{\ell_{1}=0}^{n-j+1-L_{2}^{j}}\left(\prod_{k=L_{1}^{j}+j}^{n+1} \lambda_{k 0}-\prod_{k=L_{1}^{j}+j}^{n+1} \lambda_{k 1}\right)^{n} \\
& \times(-1)^{\ell_{j}}\left[\prod_{\mu=1}^{j-1}(-1)^{\chi_{\mu} \ell_{\mu}}\right] \times\left[\prod_{\mu=1}^{j-1}\left(\prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(\chi_{\mu} \chi_{k}\right)}-\prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(1-\chi_{\mu} \chi_{k}\right)}\right)\right]
\end{aligned}
$$

Adding together yields the formula

$$
\begin{aligned}
\alpha_{j}^{(n+1)} & =\sum_{\ell_{j}=0}^{n-j+1}\left[\prod_{k=0}^{\ell_{j}} \lambda_{k \chi_{k}}^{2}\right] \sum_{\ell_{j-1}=0}^{n-j+1-L_{j}^{j} \sum_{\ell_{j-2}=0}^{n-j+1-L_{j-1}^{j}} \sum_{\ell_{1}=0}^{n-j+1-L_{2}^{j}}\left(\sum_{k=L_{1}^{j}+j}^{n+1} \lambda_{k 0}-\prod_{k=L_{1}^{j}+j}^{n+1} \lambda_{k 1}\right)^{2}} \\
& \times\left[\prod_{\mu=1}^{j}(-1)^{\chi_{\mu} \ell_{\mu}}\right] \times\left[\prod_{\mu=1}^{j-1}\left(\prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(\chi_{\mu} \chi_{k}\right)}^{n} \prod_{k=L_{1+\mu}^{j}+j-\mu}^{L_{\mu}^{j}+j-\mu} \lambda_{k,\left(1-\chi_{\mu} \chi_{k}\right)}^{2}\right)^{2}\right]
\end{aligned}
$$

The formula holds for $j$ even or odd, and so this concludes the proof.
As stated earlier in the paper, the formulas (36) and (37) are dependent on our choice of $\lambda_{j l}$; ideally, any set of $\lambda_{j l}$ such that $\overline{\lambda_{k 0}^{2}}=\lambda_{k 1}^{2}$ and the sequences preserves special unitarity should have these identities. All that remains is to prove the identities we desire, which seem likely to be true but difficult to do.

Conjecture 5.8. The above formula (37) for $\beta_{j}^{(n)}$, when plugging in roots of unity $\lambda_{j 0}=\omega^{(-1)^{j} j}$ and $\lambda_{j 1}=(-1)^{j+1} \omega^{(-1)^{j+1} j}$, for $\omega=e^{i \theta / 2}$ and $\theta=\frac{\pi}{2 n+1}$, yield the relation $\beta_{1}^{(n)}=-(-1)^{\frac{n(n+3)}{2}}, \beta_{2 i}^{(n)}=$ $(-1)^{\frac{n(n+3)}{2}}\binom{n-i}{n-2 i}$, and $\beta_{2 i+1}^{(n)}=-(-1)^{\frac{n(n+3)}{2}}\binom{n-i-1}{n-2 i-1}$ for $i \in \mathbb{N}$.

This conjecture appears remarkably hard to prove and will likely be much harder to show than the contents of this paper. This question will remain open, hopefully to be proven in a later paper. However, if this conjecture is true, it would directly imply Conjecture 2.1 provided the validity of the arguments in this paper.
5.2. Applications to $\mathbf{n}=\mathbf{3}$. Here we consider a special case of the formulas above, namely $n=3$, which would correspond to the angle $\theta=\pi / 7$. We already know that $\beta_{0}^{(3)}=-1, \beta_{1}^{(3)} / \beta_{0}^{(3)}=-1$ and $\beta_{3}^{(3)} / \beta_{0}^{(3)}=$ $(-1)^{3}=-1$ as desired, but $\beta_{2}^{(3)} / \beta_{0}^{(3)}$ has not been examined yet. This special case is quite easy to show. Taking the formulas (36) and (37) and plugging in $j=2$ yields the expressions

$$
\begin{equation*}
\alpha_{2}^{(n)}=\sum_{\ell_{2}=0}^{n-2}\left[\prod_{k=0}^{\ell_{2}} \lambda_{k 0}^{2}\right] \sum_{\ell_{1}=0}^{n-2-\ell_{2}}(-1)^{\ell_{1}}\left(\prod_{k=\ell_{2}+1}^{\ell_{1}+\ell_{2}+1} \lambda_{k \chi_{k}}-\prod_{k=\ell_{2}+1}^{\ell_{1}+\ell_{2}+1} \lambda_{k \chi_{k+1}}\right)^{2}\left(\prod_{k=\ell_{1}+\ell_{2}+2}^{n} \lambda_{k 0}-\prod_{k=\ell_{1}+\ell_{2}+2}^{n} \lambda_{k 1}\right)^{2} \tag{38}
\end{equation*}
$$

$$
\begin{equation*}
\beta_{2}^{(n)}=\sum_{\ell_{2}=0}^{n-2}\left[\prod_{k=0}^{\ell_{2}} \lambda_{k 0} \lambda_{k 1} \sum_{\ell_{1}=0}^{n-2-\ell_{2}}(-1)^{\ell_{1}}\left(\prod_{k=\ell_{2}+1}^{\ell_{1}+\ell_{2}+1} \lambda_{k \chi_{k}}-\prod_{k=\ell_{2}+1}^{\ell_{1}+\ell_{2}+1} \lambda_{k \chi_{k+1}}\right)^{2}\left(\prod_{k=\ell_{1}+\ell_{2}+2}^{n} \lambda_{k 0}-\prod_{k=\ell_{1}+\ell_{2}+2}^{n} \lambda_{k 1}\right)^{2}\right. \tag{39}
\end{equation*}
$$

Plugging in $n=3$ yields the expressions

$$
\begin{gathered}
\alpha_{2}^{(3)}=\left(\lambda_{11}-\lambda_{10}\right)^{2}\left(\lambda_{20} \lambda_{30}-\lambda_{21} \lambda_{31}\right)^{2}-\left(\lambda_{11} \lambda_{20}-\lambda_{10} \lambda_{21}\right)^{2}\left(\lambda_{30}-\lambda_{31}\right)^{2}+\lambda_{10}^{2}\left(\lambda_{20}-\lambda_{21}\right)^{2}\left(\lambda_{30}-\lambda_{31}\right)^{2} \\
\beta_{2}^{(3)}=\left(\lambda_{11}-\lambda_{10}\right)^{2}\left(\lambda_{20} \lambda_{30}-\lambda_{21} \lambda_{31}\right)^{2}-\left(\lambda_{11} \lambda_{20}-\lambda_{10} \lambda_{21}\right)^{2}\left(\lambda_{30}-\lambda_{31}\right)^{2}+\lambda_{10} \lambda_{11}\left(\lambda_{20}-\lambda_{21}\right)^{2}\left(\lambda_{30}-\lambda_{31}\right)^{2}
\end{gathered}
$$

Here we are mainly concerned with $\beta_{2}^{(3)}$ and the value it takes.
Lemma 5.9. For $\theta=\pi / 7$,

$$
\beta_{2}^{(3)}=(2 \cos \theta-2)(2 \cos \theta+2)-(2 \cos 3 \theta-2 \cos 2 \theta)(2 \cos 3 \theta-2)=-2
$$

Proof. To start, $2 \cos 3 \theta-2 \cos 2 \theta=1-2 \cos \theta$ by Lemma A.1, and $(2 \cos \theta-2)(2 \cos \theta+2)=2 \cos 2 \theta-2$ by double angle identities.

$$
\begin{aligned}
& \beta_{2}^{(3)}=(2 \cos 2 \theta-2)-(1-2 \cos \theta)(2 \cos 3 \theta-2)=-4 \cos \theta+2 \cos 2 \theta-2 \cos 3 \theta+4 \cos \theta \cos 3 \theta \\
& \quad=-4 \cos \theta+2 \cos 2 \theta-2 \cos 3 \theta+2 \cos 2 \theta+2 \cos 4 \theta=-4 \cos \theta+4 \cos 2 \theta-4 \cos 3 \theta=-2
\end{aligned}
$$

From these results we get that $\left(\beta_{0}^{(3)}, \beta_{1}^{(3)}, \beta_{2}^{(3)}, \beta_{3}^{(3)}\right)=(-1,1,-2,1)$. This matches our conjecture as $\beta_{1}^{(3)}=-\beta_{0}^{(3)}, \beta_{2}^{(3)}=\beta_{0}^{(3)}\binom{3-1}{3-2}=-2$, and $\beta_{3}^{(3)}=-\beta_{0}^{(3)}\binom{3-1-1}{3-2-1}=1$. This means that

$$
\begin{aligned}
\mathcal{B}_{k}^{(3)} & =\beta_{0}+\left|a_{k}\right|^{2}\left(\beta_{1}-\left|b_{k}\right|^{2}\left(\beta_{2}+\left|a_{k}\right|^{2} \beta_{3}\right)\right) \\
& =\beta_{0}+\left|a_{k}\right|^{2}\left(\beta_{1}+\beta_{2}\left(-\left|b_{k}\right|^{2}\right)+\beta_{3}\left(-\left|b_{k}\right|^{2}+\left|b_{k}\right|^{4}\right)\right) \\
& =-1+\left|a_{k}\right|^{2}\left(1-2\left(-\left|b_{k}\right|^{2}\right)+\left(-\left|b_{k}\right|^{2}+\left|b_{k}\right|^{4}\right)\right) \\
& =-1+\left(1-\left|b_{k}\right|^{2}\right)\left(1+\left|b_{k}\right|^{2}+\left|b_{k}\right|^{4}\right)=-1+\left(1-\left|b_{k}\right|^{6}\right)=-\left|b_{k}\right|^{6}
\end{aligned}
$$

Therefore we get $\left|\left(U_{k+1}^{(3)}\right)_{21}\right|=\left|b_{k}^{(3)} \mathcal{B}_{k}^{(3)}\right|=\left|b_{k}^{(3)}\right| \cdot\left|b_{k}^{(3)}\right|^{6}=\left|b_{k}^{(3)}\right|^{7}$, which is our desired relation. This may appear to be a roundabout approach to simply expanding the sequence directly, but this skips the step of having to combine the matrices and write the sequence in a convenient manner. By multiplying by the constant $\left(\alpha_{0}^{(3)}\right)^{-1}$ and writing $D(\theta)=\operatorname{diag}\left(1, e^{i \theta}\right)$ we can also construct the sequence

$$
\begin{equation*}
U_{k+1}=U_{k} D(\theta) U_{k}^{-1} D(\theta)^{5} U_{k} D(\theta)^{3} U_{k}^{-1} D(\theta)^{3} U_{k} D(\theta)^{5} U_{k}^{-1} D(\theta) U_{k} \tag{40}
\end{equation*}
$$

This is because we distribute a $\lambda_{j 0}^{-1}$ to each $D_{j}$, giving $D_{j} \lambda_{j 0}^{-1}=\operatorname{diag}\left(1,(-1)^{j+1} e^{i \theta_{p}(-1)^{j+1} j}\right)$, so $D_{1} \lambda_{10}^{-1}=$ $D(\theta), D_{2} \lambda_{20}^{-1}=D(\theta)^{5}$, and $D_{3} \lambda_{30}^{-1}=D(\theta)^{3}$ by using the identity $e^{j \pi i}=(-1)^{j}$. By the convergence of $U_{k+1}^{(3)}$, since these sequences differ by an additional phase multiplied at the end, they both converge the same way.
5.3. Additional Considerations. In order for these sequences to actually converge to a single diagonal matrix, we must apply one modification, due to the choice of $\lambda_{j l}$ we made. Recall that the leading term in $\mathcal{A}_{k}$ is $\alpha_{0}=\prod \lambda_{k 0}^{2}$. As $b_{k} \rightarrow 0$, the higher order terms vanish, and $\alpha_{0}$ will only remain as the factor on $a_{k}$, but this factor means that $a_{k}$ will rotate on the unit circle endlessly, and if we don't want this, we need to apply an additional matrix to the end of our sequence. Say that our sequence is $U_{k+1}=A_{N}\left(U_{k} ; \theta\right)$, then multiply by a matrix $F=\operatorname{diag}\left(\alpha_{0}^{-1}, \alpha_{0}\right)=\operatorname{diag}\left(\prod \lambda_{k 1}^{2}, \Pi \lambda_{k 0}^{2}\right)$. So our sequence is now $U_{k+1}^{(N)}=A_{N}\left(U_{k} ; \theta\right) F$, and so our sequences are now multiplied by $\alpha_{0}^{-1}$. Since this is just a factor with magnitude 1 , this still maintains the property $\left|b_{k+1}\right|=\left|b_{k}\right|^{N}$. The benefit to this factor is that now $\mathcal{A}_{k}$ takes the form $\mathcal{A}_{k}=1-\left|b_{k}\right|^{2}(\ldots) \rightarrow 1$ as $k \rightarrow \infty$. What this means is that the sequence $\left\{U_{k}^{(n)}\right\}_{k=0}^{\infty}$ will now converge to the identity gate instead of $a_{k}^{(n)}$ approximately rotating on the complex unit circle as $k \rightarrow \infty$.

## 6. Composing Convergent Sequences for Composite Angles

Recall that we are looking for sequences of the form $U_{k+1}=A_{N}\left(U_{k} ; \theta\right)$, where $\left|b_{k+1}\right|=\left|b_{k}\right|^{N}$, with $N \geq 3$ an odd number and $\theta$ a given angle. If $N$ is composite, then there is a naïve way to find a convergent sequence, which is by taking sequence composition. By sequence composition we mean taking two convergent sequences defined by the relations $U_{k+1}^{\left(n_{1}\right)}=A_{p_{1}}\left(U_{k} ; \theta_{p_{1}}\right)$ and $U_{k+1}^{\left(n_{2}\right)}=A_{p_{2}}\left(U_{k} ; \theta_{p_{2}}\right)$ and defining a new sequence using the composition $U_{k+1}=A_{p_{2}}\left(A_{p_{1}}\left(U_{k} ; \theta_{p_{1}}\right) ; \theta_{p_{2}}\right)$. This is a very different approach to the above, but it is an effective strategy if the only desired outcome is this diagonalization.

Theorem 6.1. Provided two sequences $A_{N_{1}}\left(U_{k} ; \frac{\pi}{N_{1}}\right)$ and $A_{N_{2}}\left(U_{k} ; \frac{\pi}{N_{2}}\right)$, we can take the composition of the sequences such that $U_{k+1}=A_{N_{1} N_{2}}\left(U_{k} ; \frac{\pi}{N_{1} N_{2}}\right)$ has the property that $\left|b_{k+1}\right|=\left|b_{k}\right|^{N_{1} N_{2}}$.

Proof. Let $V=A_{N_{1}}\left(U_{k} ; \pi / N_{1}\right)$, with the off diagonal element $b=V_{21}$ having the property that $|b|=\left|b_{k}\right|^{N_{1}}$. Then we plug this into the next sequence and get $U_{k+1}=A_{N_{2}}\left(V ; \pi / N_{2}\right)$, and since $A_{N_{1}}$ preserves the unitary property we have $\left|b_{k+1}\right|=|b|^{N_{2}}=\left|b_{k}\right|^{N_{1} N_{2}}$. The only part remaining is to manipulate our sequences slightly, since each angle has to be $\theta=\frac{\pi}{N_{1} N_{2}}$. Here, let $N=N_{1} N_{2}$, and we can write our sequence as

$$
\begin{equation*}
U_{k+1}=A_{N_{2}}\left(A_{N_{1}}\left(U_{k} ; \theta \frac{N}{N_{1}}\right) ; \theta \frac{N}{N_{2}}\right) \tag{41}
\end{equation*}
$$

Plugging in $\theta=\pi / N$ yields our desired relation.
By generously applying this theorem, we can construct a prime factorization for a sequence of any composite angle.

Corollary 6.2. Let $p_{j}$ be odd primes such that for all $j=1, \ldots, \ell$, there exists a sequence such that $U_{k+1}=A_{p_{j}}\left(U_{k} ; \pi / p_{j}\right)$ converges as $\left|b_{k+1}\right|=\left|b_{k}\right|^{p_{j}}$. Then we can compose these sequences together in any order and obtain a sequence $U_{k+1}=A_{N}\left(U_{k} ; \theta\right)$, where $N=p_{1}^{m_{1}} \ldots p_{\ell}^{m_{\ell}}$ and $\theta=\frac{\pi}{N}$, where $m_{j}$ are nonnegative integers.

One way to write this formula is as follows, where $A_{p}^{m}$ is to denote function composition of $A_{p}$ with itself $m$ times (assume that the angle applied is the same every time):

$$
\begin{equation*}
U_{k+1}^{(N)}=A_{N}\left(U_{k} ; \theta\right)=A_{p_{1}}^{m_{1}}\left(A_{p_{2}}^{m_{2}}\left(\ldots\left(A_{p_{\ell}}^{m_{\ell}}\left(U_{k} ; \theta \frac{N}{p_{\ell}}\right) \ldots\right) ; \theta \frac{N}{p_{2}}\right) ; \theta \frac{N}{p_{1}}\right) \Longrightarrow\left|b_{k+1}\right|=\left|b_{k}\right|^{N} \tag{42}
\end{equation*}
$$

This corollary also has an interesting consequence: for any composite $N$, sequences of this type that converge as $\left|b_{k+1}\right|=\left|b_{k}\right|^{N}$ for $\theta=\pi / N$ are not unique up to a phase angle. The reason for this is that we can apply rotations

Constructing in this manner allows for a simple method of producing sequences for angles of composite $N$. In order to keep the same angle all the way through, you take every instance of $\theta$ in either of the individual sequences and multiply by $N / p_{j}$, that way you have a modified sequence that can converge for $\theta=\pi / N$ instead of $\pi / p_{1}$. This is quite an abstract formulation, so an example helps.

Example 6.3. Let $D(\theta)=\operatorname{diag}\left(\omega, \omega^{-1}\right)$, with $\omega=e^{i \theta / 2}$. This is a different formulation to the one we have used in the paper, but it is equivalent up to a factor by bringing the $(-1)^{j+1}$ factor into the exponential in $\lambda_{j 1}$. We can construct a sequence for $N=15$ by using the sequences $U_{k+1}=A_{3}\left(U_{k}\right)=U_{k} D(\theta) U_{k}^{-1} D(\theta) U_{k}$ and $U_{k+1}=A_{5}\left(U_{k}\right)=U_{k} D(\theta) U_{k}^{-1} D(\theta)^{3} U_{k} D(\theta)^{3} U_{k}^{-1} D(\theta) \cdot U_{k}$. As predicted, $A_{5}$ and $A_{3}$ take optimal angles $\theta_{5}=\pi / 5$ and $\theta_{3}=\pi / 3$ respectively. Our combined angle is $\theta=\pi / 15$. We can take the composition we defined above, letting

$$
\begin{gathered}
A_{N}\left(U_{k} ; \theta\right)=A_{5}\left(A_{3}\left(U_{k} ; \theta \cdot N / 3\right) ; \theta \cdot N / 5\right)=A_{5}\left(A_{3}\left(U_{k} ; \theta \cdot 5\right) ; \theta \cdot 3\right) \\
A_{3}\left(U_{k} ; \theta \cdot n / 3\right)=U_{k} D(\theta \cdot 5) U_{k}^{-1} D(\theta \cdot 5) U_{k}=U_{k} D(\theta)^{5} U_{k}^{-1} D(\theta)^{5} U_{k} \\
\left(A_{3}\left(U_{k} ; \theta \cdot n / 3\right)\right)^{-1}=U_{k}^{-1} D(\theta)^{-5} U_{k} D(\theta)^{-5} U_{k}^{-1} \\
A_{5}\left(U_{k} ; \theta \cdot n / 3\right)=U_{k} D(\theta)^{3} U_{k}^{-1} D(\theta)^{9} U_{k} D(\theta)^{9} U_{k}^{-1} D(\theta)^{3} U_{k}
\end{gathered}
$$

Combining everything, we get (dropping $(\theta)$ from $D(\theta)$ for brevity)

$$
\begin{aligned}
U_{k+1}=A_{N}\left(U_{k} ; \theta\right)= & U_{k} D^{5} U_{k}^{-1} D^{5} U_{k} D^{3} U_{k}^{-1} D^{-5} U_{k} D^{-5} U_{k}^{-1} D^{9} U_{k} D^{5} U_{k}^{-1} D^{5} U_{k} D^{9} \\
& \cdot U_{k}^{-1} D^{-5} U_{k} D^{-5} U_{k}^{-1} D^{3} U_{k} D^{5} U_{k}^{-1} D^{5} U_{k}
\end{aligned}
$$

If we plug in $\theta=\pi / 15$, we get exactly what we want: $\left|b_{k+1}\right|=\left|b_{k}\right|^{15}$. If we write the powers of $D(\theta)$ as they appear in an ordered list, we get $(5,5,3,-5,-5,9,5,5,9,-5,-5,3,5,5)$. In contrast, the powers as they appear given by a similar formulation yields $(1,13,3,11,5,9,7,7,9,5,11,3,13,1)$. These are not equivalent up to a factor, despite having the same convergence rate. The conjectured formulation we have used throughout the paper applies multiples of angles $m \theta$ for $m=1, \ldots, n$, whereas this example consists of applied angles $m \theta$ when $m$ divides the denominator in $\theta=\frac{\pi}{N}$. Both sequences numerically provide the same diagonalizing property. Despite yielding more or less the same end result, the process by which the sequences do so is different.

## 7. Conclusion

We have provided several results on the nature of these families of diagonalizing sequences, although a few results remain to show. This paper primarily concerned itself with the critical angles for the sequence for optimal convergence, not the analysis of the behavior in a neighborhood of the angle. These sequences are also being proposed as the optimal algorithm to the given problem, but there are certainly instances of sequences of this form which are not optimal but still appear to be convergent, for example when every matrix is $D_{1}$ for any $n$.

Some mathematical details to the nature of these sequences are to be desired. In particular, the identity in (28) is a very complicated result to show, one that to our knowledge has never been proven. A good mathematical explanation as to why these identities simplify so greatly is needed. In addition, the prime factorization argument we provided disproves uniqueness of roots $\lambda_{j l}$ for composite angles, but it is still unknown for prime angles $\pi / p$. If it is unique, then what is special about those roots?

Variants of this scheme can likely be applied for arbitrary states $|s\rangle$ and $|t\rangle$, providing a powerful and efficient quantum search algorithm with minimal usage of gates.
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## Appendix A. Trigonometric Identities

Here are all of the trigonometric identities used, as well as a proof since most of them are not commonly used.

Lemma A.1. For $\theta=\pi /(2 n+1), \forall n \in \mathbb{N}$,

$$
\sum_{k=0}^{n}(-1)^{k} \cos (k \theta)=\frac{1}{2}
$$

Proof. We can convert this sum into a well known identity:

$$
\sum_{k=0}^{n}(-1)^{k} \cos (k \theta)=\sum_{k=0}^{n} \cos (k(\theta+\pi))=\frac{\sin ((n+1)(\theta+\pi) / 2)}{\sin ((\theta+\pi) / 2)} \cos (n(\theta+\pi) / 2)
$$

$\theta+\pi=(2 n+2) \pi /(2 n+1)$, and using the identity $2 \sin \theta \cos \varphi=\sin (\theta+\varphi)+\sin (\theta-\varphi)$

$$
\begin{gathered}
=\frac{\sin \left(\frac{(n+1)^{2}}{2 n+1} \pi\right)}{\sin \left(\frac{n+1}{2 n+1} \pi\right)} \cos \left(\frac{n^{2}+n}{2 n+1} \pi\right)=\frac{1}{2 \sin \left(\frac{n+1}{2 n+1} \pi\right)}\left[\sin \left(\frac{2 n^{2}+3 n+1}{2 n+1} \pi\right)+\sin \left(\frac{n+1}{2 n+1} \pi\right)\right] \\
=\frac{1}{2 \sin \left(\frac{n+1}{2 n+1} \pi\right)}\left[\sin \left(\frac{(2 n+1)(n+1)}{2 n+1} \pi\right)+\sin \left(\frac{n+1}{2 n+1} \pi\right)\right]=\frac{1}{2}
\end{gathered}
$$

Lemma A.2. For $\theta=\pi /(2 n+1), \forall n \in \mathbb{N}$,

$$
\prod_{k=1}^{n}\left(2 \cos (k \theta)+2(-1)^{k}\right)=(-1)^{\frac{n(n+1)}{2}}
$$

Proof. This statement is equivalent to saying that for $m \geq 0$, if $n=2 m$ or $2 m+1$, we have

$$
\prod_{k=1}^{2 m}\left(2 \cos (k \theta)+2(-1)^{k}\right)=(-1)^{m}, \quad \prod_{k=1}^{2 m+1}\left(2 \cos (k \theta)+2(-1)^{k}\right)=(-1)^{m+1}
$$

For $n=2 m$, we can split this product using $2 \sin ^{2}(\theta / 2)=1-\cos (\theta)$ and $2 \cos ^{2}(\theta / 2)=1+\cos (\theta)$ :

$$
\prod_{k=1}^{2 m}\left(2 \cos (k \theta)+2(-1)^{k}\right)=\prod_{k=1}^{m}(2 \cos ((2 k-1) \theta)-2)(2 \cos (2 k \theta)+2)
$$

$$
=\prod_{k=1}^{m}-16 \sin ^{2}\left(\frac{(2 k-1) \theta}{2}\right) \cos ^{2}(k \theta)=2^{4 m}(-1)^{m} \prod_{k=1}^{m} \sin ^{2}\left(\frac{(2 k-1) \theta}{2}\right) \cos ^{2}(k \theta)
$$

It remains to just prove that

$$
\begin{gathered}
\frac{1}{2^{2 m}}=\prod_{k=1}^{m} \sin \left(\frac{(2 k-1) \theta}{2}\right) \cos (k \theta)=\prod_{k=1}^{m} \sin \left(\frac{(2 k-1) \pi}{8 m+2}\right) \cos \left(\frac{k \pi}{4 m+1}\right) \\
=\prod_{k=1}^{m} \sin \left(\frac{(2 k-1) \pi}{8 m+2}\right) \sin \left(\frac{(4 m+1-2 k) \pi}{8 m+2}\right)
\end{gathered}
$$

Reversing the product order for the second sine, we get that we can combine the products as

$$
=\prod_{k=1}^{2 m} \sin \left(\frac{(2 k-1) \pi}{8 m+2}\right)=\prod_{k=1}^{n} \sin \left(\frac{(2 k-1) \pi}{4 n+2}\right)=\frac{1}{2^{n}}=\frac{1}{2^{2 m}}
$$

As for the odd case, it is much of the same setup:

$$
\begin{gathered}
\prod_{k=1}^{2 m+1}\left(2 \cos (k \theta)+2(-1)^{k}\right)=\prod_{k=1}^{m+1}(2 \cos ((2 k-1) \theta)-2) \prod_{k=1}^{m}(2 \cos (2 k \theta)+2) \\
=2^{4 m+2}(-1)^{m+1} \prod_{k=1}^{m+1} \sin ^{2}\left(\frac{(2 k-1) \theta}{2}\right) \prod_{k=1}^{m} \cos ^{2}(k \theta)
\end{gathered}
$$

It remains to prove that

$$
\begin{gathered}
\frac{1}{2^{2 m+1}}=\prod_{k=1}^{m+1} \sin \left(\frac{(2 k-1) \theta}{2}\right) \prod_{k=1}^{m} \cos (k \theta)=\prod_{k=1}^{m+1} \sin \left(\frac{(2 k-1) \pi}{8 m+6}\right) \prod_{k=1}^{m} \cos \left(\frac{k \pi}{4 m+3}\right) \\
=\prod_{k=1}^{m+1} \sin \left(\frac{(2 k-1) \pi}{8 m+6}\right) \prod_{k=1}^{m} \sin \left(\frac{(4 m+3-2 k) \pi}{8 m+6}\right)
\end{gathered}
$$

Again, reversing the product order in the second product allows us to combine these together:

$$
=\prod_{k=1}^{2 m+1} \sin \left(\frac{(2 k-1) \pi}{8 m+6}\right)=\prod_{k=1}^{n} \sin \left(\frac{(2 k-1) \pi}{4 n+2}\right)=\frac{1}{2^{n}}=\frac{1}{2^{2 m+1}}
$$

Lemma A.3. For $n \in \mathbb{N}$ :

$$
\prod_{k=1}^{n} \sin \left(\frac{(2 k-1) \pi}{4 n+2}\right)=\frac{1}{2^{n}}
$$

Proof.

$$
\begin{gathered}
\prod_{k=1}^{n} \sin \left(\frac{(2 k-1) \pi}{4 n+2}\right)=\prod_{k=1}^{n} \cos \left(\frac{\pi}{2}-\frac{(2 k-1) \pi}{4 n+2}\right)=\prod_{k=1}^{n} \cos \left(\frac{(n+1-k) \pi}{2 n+1}\right) \\
=\prod_{k=1}^{n} \cos \left(\frac{k \pi}{2 n+1}\right) \equiv P
\end{gathered}
$$

Multiply by the sine counterpart:

$$
\begin{aligned}
& Q \equiv \prod_{k=1}^{n} \sin \left(\frac{k \pi}{2 n+1}\right), \quad P \cdot Q=\prod_{k=1}^{n} \cos \left(\frac{k \pi}{2 n+1}\right) \sin \left(\frac{k \pi}{2 n+1}\right)=\prod_{k=1}^{n} \frac{1}{2} \sin \left(\frac{2 k \pi}{2 n+1}\right) \\
= & \frac{1}{2^{n}} \prod_{k \leq \frac{n}{2}} \sin \left(\frac{2 k \pi}{2 n+1}\right) \prod_{k>\frac{n}{2}} \sin \left(\frac{2 k \pi}{2 n+1}\right)=\frac{1}{2^{n}} \prod_{k \leq \frac{n}{2}} \sin \left(\frac{2 k \pi}{2 n+1}\right) \prod_{k>\frac{n}{2}} \sin \left(\frac{(2 n+1-2 k) \pi}{2 n+1}\right)
\end{aligned}
$$

The first product covers every even numbered index, and the second handles all of the odd numbered cases, and so this reduces to

$$
=\frac{1}{2^{n}} \prod_{k=1}^{n} \sin \left(\frac{k \pi}{2 n+1}\right)=\frac{1}{2^{n}} Q \Longrightarrow P \cdot Q=\frac{1}{2^{n}} Q \Longrightarrow P=\prod_{k=1}^{n} \cos \left(\frac{k \pi}{2 n+1}\right)=\frac{1}{2^{n}}
$$

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Department of Mathematics and Department of Physics and Astronomy, Purdue University, West Lafayette, IN 47907, U.S.A.

Email address: griff254@purdue.edu
Department of Mathematics and Department of Physics and Astronomy, Purdue University, West Lafayette, IN 47907, U.S.A.

Email address: cui177@purdue.edu


[^0]:    Date: June 2021.
    ${ }^{1}$ If fact, the term $D(\theta)^{-7}$ has to be appended to the RHS of Equation 1 in order for the sequence to converge. Otherwise, it would have several convergent subsequences. This will not affect our discussions below though.

