

Quantum circuits for toric code and X-cube fracton model

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Abstract

We introduce a systematic and efficient quantum circuit only composed of Clifford gates to achieve the ground state of surface code model. Our algorithm transforms the question into a purely geometric one, which can be easily extended to achieve the ground state of some 3D topological phases i.e. 3D toric code model and X-cube fracton model. We also introduce the gluing method with measurements to enable our method to achieve the ground state of 2D toric code on an arbitrary planar lattice and pave the way to more complicated 3D topological phases.

1 Introduction

The subject of topological phases of matter (TPMs) has been under extensive study for the past few decades. Topological phases are gapped spin liquids at low temperatures which are not described by the conventional Landau theory of spontaneous symmetry breaking and local order parameters; instead, they are characterized by a new order, *topological order*. The ground states of a topological phase have stable degeneracy and robust long range entanglement. Topological phases in 2D also support quasi-particle excitations with anyonic exchange statistics which make them an appealing platform to fault-tolerantly store and process quantum information. Two peculiar features among others are that the ground state degeneracy is a topological invariant of the underlying system, and that the quasi-particles can freely move without costing energy. A large class of topological phases is realized by exactly solvable spin lattice models with bosonic degrees of freedom. A paradigmatic example in 2D is the toric code, and more generally Kitaev's quantum double model based on finite groups [6], and yet even more generally the Levin-Wen string-net model based

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on fusion categories [7]. Examples of 3D topological phases include toric code (3D version) and the Walker-Wang model based on premodular categories [18].

In recent years, more exotic phases in 3D, called fracton phases, have been discovered [5, 16, 17]. Fractons also possess stable ground state degeneracy and long range entanglement. However, the ground state degeneracy of fractons depends on the system size, and hence is not a topological invariant. Moreover, the mobility of excitations is constrained. The excitations can only move in certain subsystems or cannot move at all. Well known examples of fractons include the Haah code [5] and the X-cube model [17]. While regular topological phases are described by topological quantum field theories, it is still an open question what theories mathematically characterize fractons. Since fractons also satisfy the topological order conditions in the sense of [1], we call the ground states of a fracton topologically ordered states, in the same way as those of regular topological phases.

Realizing topological phases in physical systems remains an extremely challenging task. On the other hand, there now exist quantum processors based on a number of platforms such as superconducting qubits [10], Rydberg atomic arrays [4], etc. These devices can host physical qubits at the scale of 10^2 , and this number is expected to increase significantly in the near future. Hence, it is both feasible and interesting to simulate topological phases in quantum processors. Thanks to the intrinsic robustness of topological phases, the simulation is relatively less sensitive to the noises in the current quantum processors. We may also gain more insight in topological phases by engineering them in processors.

The toric code ground states were realized in the superconducting-qubit-based systems [10] and the Rydberg-atom systems [14]. In [10], the authors gave a quantum circuit consisting of Clifford gates to realize the ground states of the planar toric code (a.k.a. surface code [3]). Quantum circuits realizing non-Abelian topological orders such as Levin-Wen string-net model and Kitaev quantum double model have also been studied. See for instance [8, 11, 15, 2, 12, 13], though in these cases, the gates utilized are no longer in the Clifford group and measurements are required.

In this paper, we develop quantum circuits realizing the ground states for a number of topological phases. In [10], only planar toric code is considered where the lattice is defined on a planar surface. Here we generalize their method to apply to a large class of surfaces with or without boundary. The quantum circuit consists of only Clifford gates. In toric code, the Hamiltonian consists of two types of operators, the term A_v for each vertex v and the term B_p for each plaquette p . See Figure 2. The key idea of constructing the ground state in [10] is as follows. Start with the product state $|\phi_0\rangle = \otimes|0\rangle$ which is the +1 eigenstate for all vertex terms. The ground state is then obtained by projecting $|\phi_0\rangle$ to the +1 eigenstate of all plaquette operators, that is,

$$|GS\rangle \sim \prod_p \frac{1+B_p}{2} |\phi_0\rangle. \quad (1)$$

The effect of $\frac{1+B_p}{2}$ acting on certain states can be simulated by an appropriate

combination of the Hadamard gate and the CNOT gate. For this method to work, the control qubit for CNOT has to be in the $|0\rangle$ state prior to applying the Hadamard and CNOT. Hence, it is critical to choose the right sequence for the plaquettes so that, immediately before simulating the term corresponding to each plaquette p , there is always at least one edge on the boundary of p with the state $|0\rangle$. When the lattice is a simple planar lattice, the problem can be easily solved by dividing the lattice into several parts and applying the CNOT gates in a specific order. In this paper, since we consider lattices on arbitrary surfaces, this question is much subtler.

Here we provide an explicit algorithm to determine the sequence in which the plaquette operators are simulated. We show that this is always possible for a large class of lattices with or without boundary. The result of the algorithm is a quantum circuit consisting of Clifford gates realizing the ground state of the toric code. Moreover, we also adapt this method to 3D phases including the 3D toric code and the X-cube fracton model. For the X-cube model, we again initialize the state to the product of $|0\rangle$ state and simulate the projectors corresponding to cube terms. A similar issue arises that we need to choose the correct sequence to simulate the cube terms. We note that the circuit we provide here realizes an *exact* ground state of the X-cube model. By comparison, using cluster states and measurements, the authors in [15] gave an *approximate* realization of the model.

In addition to the above method using only quantum gates, we also provide a different way of realizing the same states. The alternative way, called gluing method, combines Clifford gates and measurement of the Pauli X gate. The resulting circuit has a shorter depth than the first one. Of course, for the toric code or X-cube, it is possible to only use measurement to obtain the ground state. Considering that frequent measurements in near-term quantum processors are costly, our method is a trade-off between circuit depth and degree of measurements.

2 Realizing ground state of 2D toric code

2.1 Toric code

It is well known that for any Hamiltonian in the form

$$H = - \sum_i H_i, \quad (2)$$

where all elements in H_i are projectors and mutually commuting, $|GS\rangle$ is the ground state as long as it is non-zero:

$$|GS\rangle = \prod_i H_i |\phi\rangle, \quad (3)$$

where $|\phi\rangle$ stands for an arbitrary state. Specifically, in a given connected lattice Γ , V refers to the set of vertices, P refers to the set of plaquettes and E refers

to the set of edges. We define $Bo(p) \subseteq E, p \in P$ to represent the border edges of the plaquette p , $\tau(e) \subseteq P, e \in E$ to represent the plaquettes consist of the edge e , and $\sigma(e) \in V, e \in E$ represents the vertices attached with the edge e . As each edge is associated with a qubit, we may abuse the notation $e, e \in E$ to represent the qubit attached. As an example, if an edge e appears as a subscript of an operator, it means the operator acts on the qubit attached to the edge e .

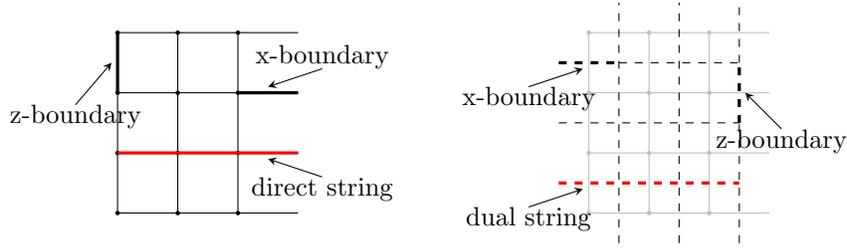


Figure 1: The black solid net on left represents the lattice Γ and the black dashed net on right represents the dual of Γ induced by the gray net.

As shown in Figure 1, an edge e is a z -boundary when $\tau(e)$ contains only one element, and it is an x -boundary if $\sigma(e)$ contains only one element (see [3] for details). On the lattice Γ , a direct string S is a series of edges $e_i, i = 1 \cdots n$ such that $\tau(e_i) \cap \tau(e_{i+1}) \neq \emptyset$ for $1 \leq i < n$. A direct string operator $F(S)$ is one operator applies X on all edges along the string S and creates two electric charges on both ends. Similarly we can define a dual string S' , which is a direct string in the dual lattice of Γ . A dual ribbon operator $F(S')$ applies Z on all edges crossed by the dual string S' and creates two magnetic charges on both ends. Notice those dual string operators which come across a z -boundary at one end will would create (or annihilate) a magnetic charge at the other end.

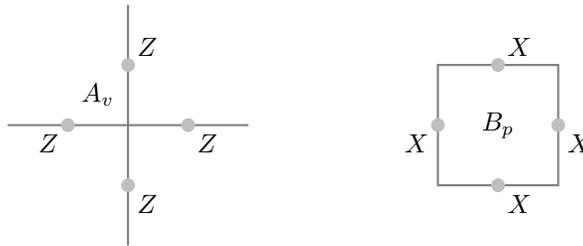


Figure 2: Definition of A_v and B_p operator in toric code.

The toric code Hamiltonian H has two terms as defined in Figure 2:

$$H = - \sum_{v \in V} A_v - \sum_{p \in P} B_p. \quad (4)$$

The action of A_v (vertex term) is to apply Pauli matrix Z over edges e if $v \in \sigma(e)$, and B_p (plaquette term) acts to apply Pauli matrix X over edges e if $p \in Bo(e)$. Naturally, we have a ground state

$$|GS\rangle = \prod_{p \in P} \frac{1 + B_p}{2} |\phi_0\rangle, \quad (5)$$

where $|\phi_0\rangle$ is the product state with each qubit set to be $|0\rangle$. It is non-zero because each component has a positive coefficient.

2.2 Single plaquette

To systematically introduce our method to simulate the ground state, we begin with the simplest case: applying $\frac{1+B_p}{2}$ on a single plaquette, which is the *basic structure* in 2D toric code. A Hadamard gate H is naturally described by $\frac{X+Z}{\sqrt{2}}$, and CNOT gate $C_{i \rightarrow j}$ is defined as

$$C_{i \rightarrow j} |ij\rangle = \frac{1 - Z_i}{2} X_j + \frac{1 + Z_i}{2} |ij\rangle, \quad (6)$$

where i is the control qubit and j is the target qubit.

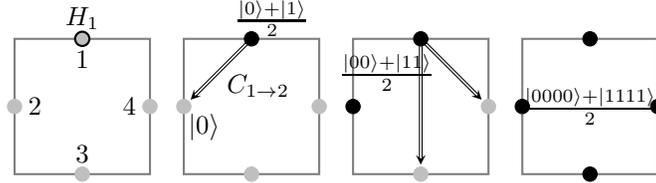


Figure 3: A qubit $|0\rangle$ is placed at each gray dot at beginning, and the color changes to black when a quantum gate is applied on the qubit; The circle on a dot means applying Hadamard gate on the qubit and an arrow stands for a CNOT gate pointing from control qubit to target qubit.

In the single plaquette shown in Figure 3, we have four qubits denoted by 1, 2, 3, 4 and we set all qubits at $|0\rangle$ as the initial state. We will apply Hadamard and CNOT gates in a specific order described in the figure. After applying H_1 and $C_{1 \rightarrow 2}$, we will have

$$C_{1 \rightarrow 2} H_1 |0000\rangle = \left(\frac{1 - Z_1}{2} X_2 + \frac{1 + Z_1}{2} \right) \frac{X_1 + Z_1}{\sqrt{2}} |0000\rangle = \frac{X_1 X_2 + 1}{\sqrt{2}} |0000\rangle. \quad (7)$$

Applying the other CNOT gates results in

$$\prod_{i=2}^4 C_{1 \rightarrow i} H_1 |0000\rangle = \frac{X_1 X_2 X_3 X_4 + 1}{\sqrt{2}} |0000\rangle = \frac{1 + B_p}{\sqrt{2}} |0000\rangle, \quad (8)$$

which is the desired ground state. Notice this procedure works as long as there exists a qubit from $Bo(p)$ in $|0\rangle$ at the beginning. We call the qubit *free qubit*, and the existence of them is crucial when we think about many plaquettes cases.

2.3 Developing to a surface with boundary

Given a complicated lattice Γ at $|\phi_0\rangle$, we need to find a path through all plaquettes p_i , $\bigcup_i p_i = P$ with a series of edges $e_i \in Bo(p_i)$ such that $e_i \notin \bigcup_{j=1}^{i-1} p_j$. Then we can use e_i as the free qubit to apply a series of operators on the introduced basic structure to achieve $\prod_i \frac{1+B_{p_i}}{\sqrt{2}}$ over $|0 \cdots 0\rangle$, which is the ground state of the toric code on lattice Γ . To illustrate the procedure, we take four plaquettes as an example:

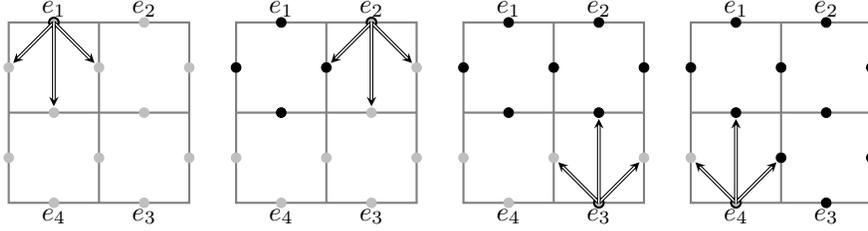


Figure 4: The procedure on the basics structure is applying Hadamard gate on any qubit at $|0\rangle$ first and CNOT gates to other qubits in any order.

As shown in Figure 4, we have chosen a path with four free qubits e_1 to e_4 . We repeat the basic structure plaquette by plaquette. Notice e_i is always $|0\rangle$ at the beginning of each step. Finally, we will get the desired ground state when the path is completed.

2.4 Developing to a surface without boundary

The situation changes for a surface without boundary. The beginning state is still $|\phi_0\rangle$, but we can not find a path with enough free qubits completing the lattices. Fortunately, because every edge sides two plaquettes, we must have

$$\prod_{p \in P} B_p = 1, \quad (9)$$

which implies we can choose a B_p to be redundant. Then we choose the last plaquette to be redundant and the path stops here. We take the lattices on a

torus as an example shown in Figure 5. We do not need to apply Hadamard gate and CNOT gates on the bottom left plaquette as we have already simulated the ground state for toric code.

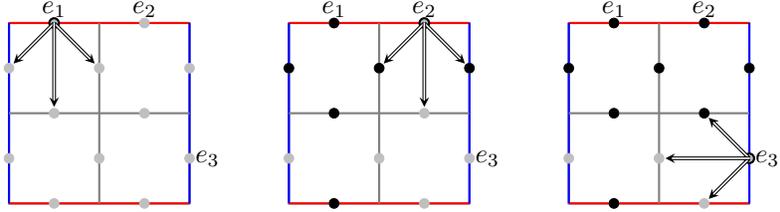


Figure 5: Boundaries with the same color are identified together to represent a genus 1 torus.

This method could be applied to a more complicated 2D surface with or without boundary as long as we can find the path. Appendix A and B show more examples and the gluing method in Section 3 will enable us to simulate ground state on arbitrary planar lattice.

2.5 Simulate arbitrary ground state

According to [6], the degeneracy of ground states for 2D toric code on torus is four: $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$. The ground state $|00\rangle$ introduced in Section 2.4 is simulated from the beginning state ϕ_0 . Because logical operators could interchange ground states and commute with B_p , we can apply them on ϕ_0 to get the other ground states.

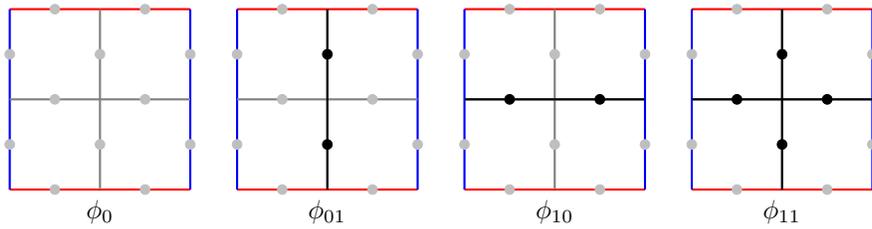


Figure 6: A qubit $|0\rangle$ is placed at each gray dot and the color changes to black when operator X flips the qubit from $|0\rangle$ to $|1\rangle$.

As shown in Figure 6, a vertical loop and a horizontal loop of X represent the two logical operators. They could change ϕ_0 into ϕ_{01} , ϕ_{10} and ϕ_{11} , which are corresponding initial states for $|01\rangle$, $|10\rangle$ and $|11\rangle$. After that, we repeat the

same procedure used in Section 2.4 but apply $X_i C_{i \rightarrow j} X_i$ instead of $C_{i \rightarrow j}$ when we encounter the flipped qubit e_i .

One step further, to get an arbitrary ground state $\phi = ae^{i\theta_a}|00\rangle + be^{i\theta_b}|01\rangle + ce^{i\theta_c}|10\rangle + de^{i\theta_d}|11\rangle$, we can apply the unitary operator U in Equation 10 on an adjacent pair of vertical and horizontal edges of ϕ_0 and use CNOT gates to transmit vertically and horizontally. After that, we just need to repeat the method above but avoid the used qubits.

$$U_1 = \begin{pmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} & 0 & 0 \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} & 0 & 0 \\ 0 & 0 & \frac{c}{\sqrt{c^2+d^2}} & \frac{-d}{\sqrt{c^2+d^2}} \\ 0 & 0 & \frac{d}{\sqrt{c^2+d^2}} & \frac{c}{\sqrt{c^2+d^2}} \end{pmatrix} \begin{pmatrix} \sqrt{a^2+b^2} & 0 & -\sqrt{c^2+d^2} & 0 \\ 0 & \sqrt{a^2+b^2} & 0 & -\sqrt{c^2+d^2} \\ \sqrt{c^2+d^2} & 0 & \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{c^2+d^2} & 0 & \sqrt{a^2+b^2} \end{pmatrix},$$

$$U_2 = \begin{pmatrix} e^{i\theta_a} & 0 & 0 & 0 \\ 0 & e^{i\theta_b} & 0 & 0 \\ 0 & 0 & e^{i\theta_c} & 0 \\ 0 & 0 & 0 & e^{i\theta_d} \end{pmatrix} \quad \text{and} \quad U = U_2 U_1. \quad (10)$$

3 Gluing method for 2D toric code

3.1 Gluing method for two single plaquettes

The method introduced in previous sections is efficient but relies on the choice of a path. It could be hard for a complicated surface, so we propose a gluing method to overcome this difficulty. We will begin with a simple example to illustrate the spirit of gluing method. To simulate the ground state of toric code on the two plaquettes in Figure 7, we can introduce an ancilla qubit to divide it into two independent plaquettes p_1 and p_2 . Let us denote the edges in $Bo(p_1)$ by 1, 2, 3, 4 and edges in $Bo(p_2)$ by 5, 6, 7, 8. We begin with ϕ_0 and ignore the overall normalization constant to simplify the calculation in the following.

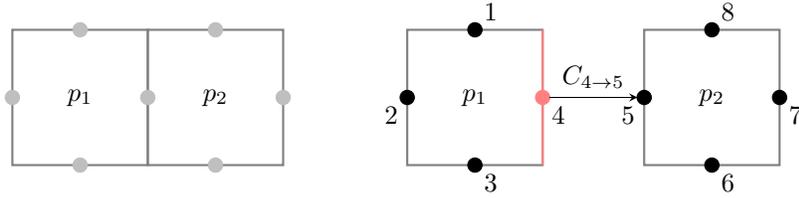


Figure 7: The lattice of two plaquettes is divided into two independent plaquettes by introducing the ancilla qubit in red.

First, apply $1 + B_{p_1}$ and $1 + B_{p_2}$ independently to get

$$(1 + X_1 X_2 X_3 X_4)(1 + X_5 X_6 X_7 X_8)|0\rangle^{\otimes 8}. \quad (11)$$

Next, apply $C_{4 \rightarrow 5}$ as and notice this operator commutes with $1 + B_{p_2}$:

$$\begin{aligned} & \left(\frac{1 - Z_4}{2} X_5 + \frac{1 + Z_4}{2} \right) (1 + X_1 X_2 X_3 X_4) (1 + X_5 X_6 X_7 X_8) |0\rangle^{\otimes 8} \\ &= (1 + X_1 X_2 X_3 X_4 X_5) (1 + X_5 X_6 X_7 X_8) |0\rangle^{\otimes 8}. \end{aligned} \quad (12)$$

Then, do a measurement over the ancilla qubit with basis $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$. If we get $+1$, it is equivalent to applying $\frac{1 + X_4}{2}$ and thus

$$\begin{aligned} & \frac{1 + X_4}{2} (1 + X_1 X_2 X_3 X_4 X_5) (1 + X_5 X_6 X_7 X_8) |0\rangle^{\otimes 8} \\ &= \left(\frac{1 + X_4}{2} \right) (1 + X_1 X_2 X_3 X_5) (1 + X_5 X_6 X_7 X_8) |0\rangle^{\otimes 8}. \end{aligned} \quad (13)$$

The ancilla qubit is disentangled now and the remaining part is precisely the ground state of two plaquettes. Observing that, when we glue two boundaries e_i and e_j together, all plaquettes terms commute with each other and $C_{i \rightarrow j}$ commutes with all plaquette terms except $1 + B_{p_k}$ where $e_i \in Bo(p_k)$. So the combination is still a ground state even if we glue multiple plaquettes together at the same time.

On the other hand, if we get -1 , it is equivalent to applying $\frac{1 - X_4}{2}$ and thus

$$\begin{aligned} & \frac{1 - X_4}{2} (1 + X_1 X_2 X_3 X_4 X_5) (1 + X_5 X_6 X_7 X_8) |0\rangle^{\otimes 8} \\ &= \left(\frac{1 - X_4}{2} \right) (1 - X_1 X_2 X_3 X_5) (1 + X_5 X_6 X_7 X_8) |0\rangle^{\otimes 8}, \end{aligned} \quad (14)$$

which is not the expected ground state. Notice it is an excited state that a magnetic charge exists at p_1 . Fortunately, we can correct it by applying Z_1 , Z_2 or Z_3 , which is a short dual ribbon operator. In the next section, we will prove that correcting operator always exists for any planar lattice.

3.2 Gluing method for many arbitrary plaquettes

When we generate the gluing method from two single plaquettes to many arbitrary plaquettes, we do not need to worry about the edges measured $+1$ but need to find a systematic method to correct the edges measured -1 . For the example shown in Figure 8, if we apply $C_{i \rightarrow j}$ to glue two boundaries and get -1 after measuring qubit e_i , the correcting operator needs to anti-commute with B_{p_i} and commute with everything else¹. A natural thought is to apply a dual string operator starting at p_i and ending crossing a z -boundary.

One step further, for any connected planar lattice $\gamma = \bigcup_{i=1}^n p_i$ with at least one z -boundary e_0 . Take a series $T = \{p_i\}$, $Bo(p_i) \cap \bigcup_{j=1}^{i-1} Bo(p_j) \neq \emptyset$ for any $i \in [2, n]$, we can insert ancilla qubits to separate it into several plaquettes and glue them together. To illustrate the idea, let us present an example consist of four plaquettes shown in Figure 9:

¹It may be easily ignored, this correcting operator commutes with all vertex terms.

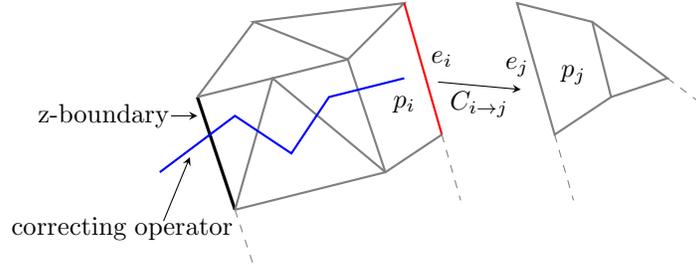


Figure 8: The bold black edge is a z-boundary and the blue line represents the dual string operator used to correct ground state if the red ancilla edge e_i is measured to be -1 after gluing.

First, we use ancilla qubits to split the lattice into single plaquettes. For $\tau(e_k)$ containing p_i and p_j , where $1 \leq i < j \leq n$ according to the series T , insert an ancilla edge e'_k into p_i and leave e_k in p_j . Then we apply $1 + B_p$ to every single plaquette $p \in P$. After that, we glue them together piece by piece. For $p_i, 1 < i \leq n$, we need to apply $C_{e' \rightarrow e}$ to all pairs of $e' \in \bigcup_{j=1}^{i-1} Bo(p_j)$ and $e \in Bo(p_i)$. Finally, we measure and disentangle e' . If we get -1, apply a dual string operator connecting p_i and the z-boundary of p_1 to correct it. Notice that we can glue all plaquettes together simultaneously and two magnetic charges could be annihilated by a dual string operator connecting them. For this example, if we get -1 for e'_1 and e'_4 simultaneously, the correcting operator will cancel out.

For the lattice without boundary, we can choose a plaquette p to be redundant and thus $Bo(p)$ become z-boundaries. Then the situation is exactly the same with the lattice with boundaries, which is left to readers. If the lattice only contains x-boundaries, we could consider the dual lattice of it, and everything is the same as flipping the plaquette and vertex operators. Thus we conclude our method is able to simulate ground state for toric code on any planar lattice.

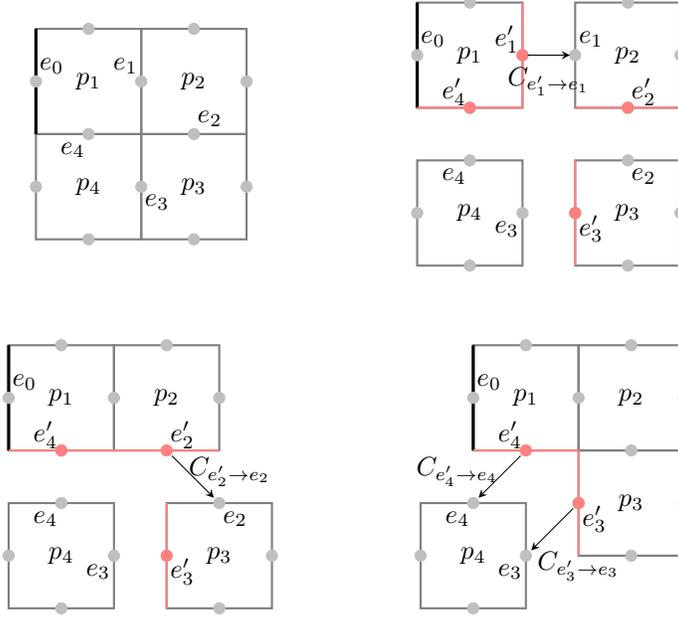


Figure 9: e_0 is a z-boundary and e' in red represents an ancilla qubit.

4 Simulate ground state for 3D models

4.1 3D toric code

We can generalize the method of 2D toric code to 3D toric code with boundary directly using a plaquette as the basic structure. It is complicated but direct, so we leave it in Section C. However, this method does not work on 3D toric code without boundary as there is no free edge in final step. To avoid this problem, we need to use a different basic structure as shown in Figure 10. We change the beginning state from $|\phi_0\rangle$ to $|\phi_{-}\rangle$, where $|-\rangle$ is set at each edge. And we need to simulate $\frac{1+A_v}{2}$ rather than $\frac{1+B_p}{2}$ to get ground state.

Using this basic structure to develop the lattice vertex by vertex, we will end with a redundant vertex as

$$\prod_{v \in V} A_v = 1. \quad (15)$$

An example consisting of eight cubes is shown in Figure 11 to illustrate the method, where the opposite faces are identified together. We begin with $|\phi_{-}\rangle$ and choose four free qubits in the lower layer to take the procedure in basic structure. After this step and identification of opposite faces, we get the lattice

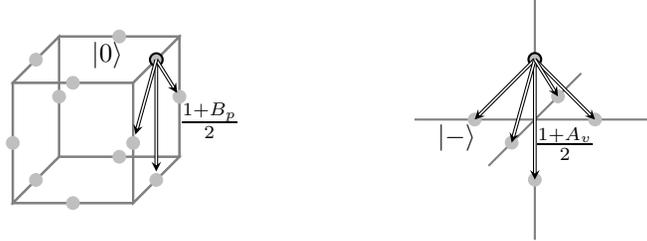


Figure 10: Comparison of two different basic structures.

with the middle untouched. Finally, choose three more free qubits to repeat the basic structure and leave a vertex redundant.

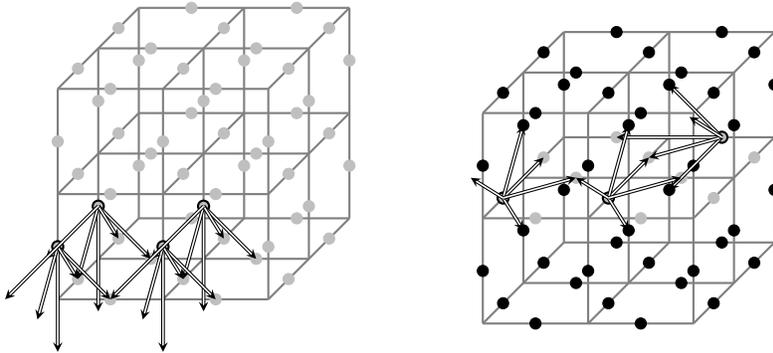


Figure 11: A qubit $|-\rangle$ is placed at each gray dot at beginning. The color changes to black when a quantum gate is applied on the qubit.

4.2 X cube code

To simulate ground state for X cube code, a model consisting of n^3 cubes with the opposite faces identified is shown in Figure 12. The beginning state is $|\phi_0\rangle$, but the basic structure is a cube rather than a plaquette. The choice of a path is obvious if we choose redundant cubes in a specific way, namely the three edges of cubes in blue. The redundancy of the blue cube in the front face comes from the identity requirement of the front layer in green. Applying this condition layer by layer, we will get a edge consist of redundant cubes. As there are three directions to slice layers independently, the desired structure ² is formed.

²There could be more redundant cubes but we only need to use the chosen ones in the method.

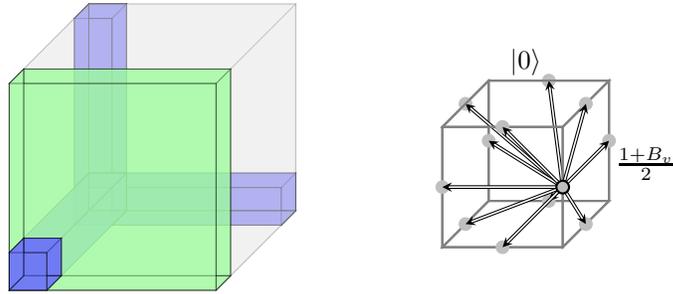


Figure 12: The left figure is a X cube model consist of n^3 cubes; The right figure shows the basic structure to be used.

To illustrate the method, we take the eight cubes case in Figure 13 as an example. Considering the redundant cubes in blue, we only need to develop four cubes left. Let us begin with the cube at the right front higher corner to apply the basic structure. After this step and identifying opposite faces, we get the result on the right-hand side of Figure 13. Then we choose three more free qubits from each cube connecting with the developed cube to repeat the procedure of basic structure. Finally, we get the ground state of X cube model consisting of eight cubes.

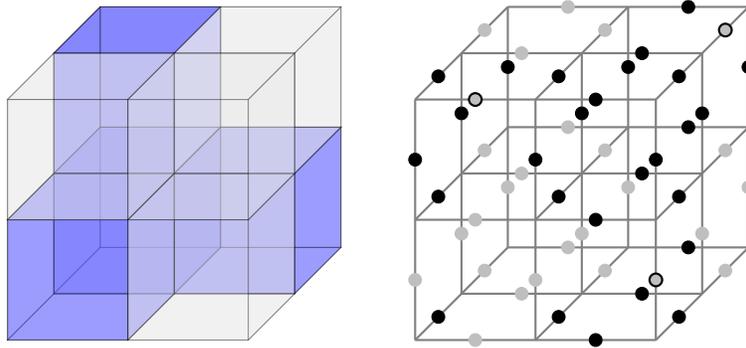


Figure 13: The left figure is an example of X cube model with opposite faces identified. The right figure shows the result after the first step and the free qubits for next step are circled.

4.3 Gluing method for 3D models

Similar to the 2D toric code case, we can simulate the ground state by breaking the lattice into basic structures, simulating on and gluing them back in the 3D toric code case. We will have one vertex term redundant and the excitations are quasi-particles that are able to move freely. The situation is exactly the same with 2D toric code, so we can find correcting operators to annihilate all of the excitations, which is left to readers. But the X-cube model is rather difficult as the excitations are fractons. The systematic method to find correcting operators is based on the following two facts:

1. There are three columns of redundant cubes as shown in Figure 12.
2. Excitation betraying cube terms is a fracton that are not able to move freely. While a *membrane operator* (see [9] for details) creates fractons on four corners of a rectangular.

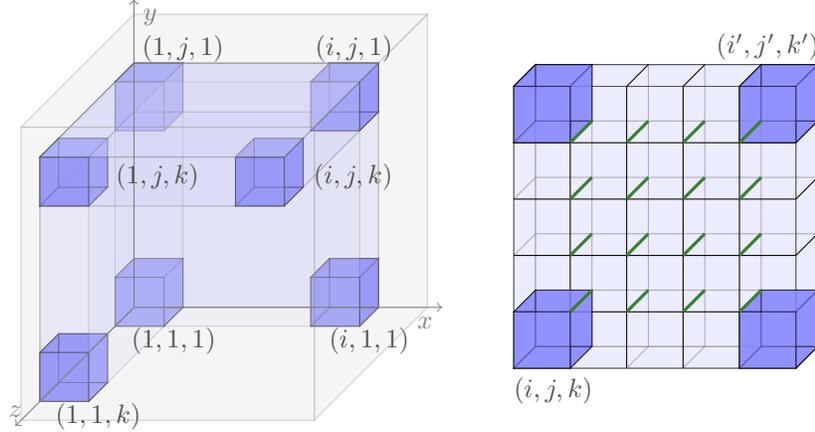


Figure 14: A membrane operator consisting of Z operators on green edges creates fractons at four corners; The correcting operator is a product of three membrane operators.

As shown in Figure 14, we label each cube by Cartesian coordinates (i, j, k) , $1 \leq i, j, k \leq n$ in a n^3 cubic lattice underlying 3D torus topology. The redundant cubes are set at three columns $(i, 1, 1)$, $(1, i, 1)$ and $(1, 1, i)$, $i = 1 \cdots n$. A membrane operator $\mathcal{M}[(i, j, k), (i', j', k')]$, consists of Z operators in the rectangle from (i, j, k) to (i', j', k') creates excitations at the four corners. If we have an excitation at $(i, j, 1)$, we can apply $\mathcal{M}[(1, 1, 1), (i, j, 1)]$ to annihilate the excitation but create excitations at redundant cubes which does not matter. For a general excitation at (i, j, k) , $i, j, k \neq 1$. We can first apply $\mathcal{M}[(1, j, 1), (i, j, k)]$ to annihilates it but creates three more at $(1, j, 1)$, $(1, j, k)$ and $(i, j, 1)$. The first one at a redundant cube can be neglected. While the other two will be annihilated

by $\mathcal{M}[(1, 1, 1), (1, j, k)]$ and $\mathcal{M}[(1, 1, 1), (i, j, 1)]$ as introduced. Thus the product operator $\mathcal{M}[(1, 1, 1), (1, j, k)] \mathcal{M}[(1, 1, 1), (i, j, 1)] \mathcal{M}[(1, j, 1), (i, j, k)]$ annihilates the general excitation.

5 Conclusion and outlook

In this paper, we use only Clifford gates to simulate ground states of 2D toric code on different surfaces with or without boundary, 3D toric code and X-cube model. We introduced the definitions of free qubit and basic structure to describe the algorithm geometrically in different models. In addition to the method using only Clifford gates, we also provide the gluing method using measurements. These two methods can be combined to simulate ground states of 2D toric code on an arbitrary surfaces or a practical system with a balance between circuit depth and degree of measurements.

There are several future directions to proceed. Naturally, we can generate our method to other 3D models of interest. Secondly, our method could be applied to non-abelian Kitaev model.

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A 2D toric code on sphere

Similar with the example of genus 1 torus, we identify different qubit pairs to change the four plaquettes into a sphere as shown in Figure 15. The bottom right plaquette is chosen to be redundant and two steps will complete the procedure.

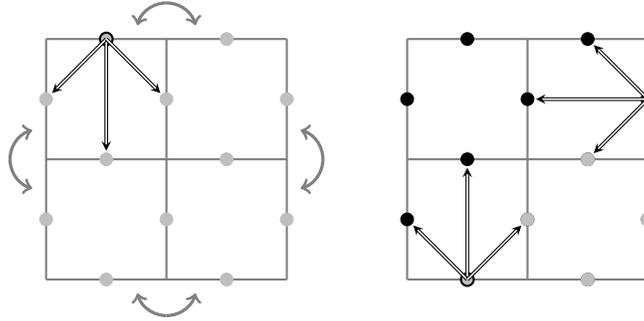


Figure 15: Boundary edges are identified according to the double-headed arrows.

B 2D toric code on genus n surface

Figure 16 shows a genus n surface which is a disk enclosed by a ribbon with identified edges. Beginning with $|\phi_0\rangle$, we develop a disk from inside and leave the ribbon with all identified edges undeveloped. Then we choose one edge in the ribbon to apply the method of basic structure and repeat in clockwise direction. After $2n - 1$ steps for a genus n torus, we will get the ground state of the closed surface.

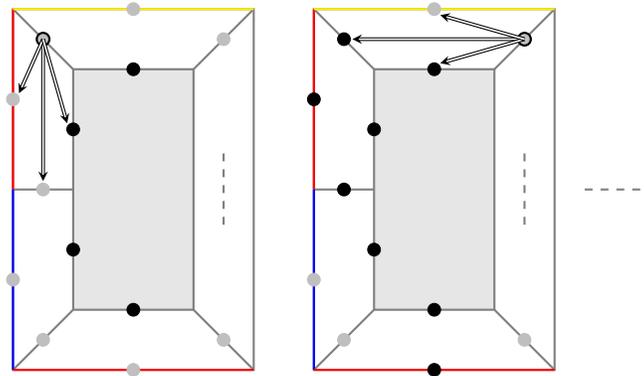


Figure 16: The shaded area represents the developed disk; Boundaries with the same color are identified to change the plaquettes into a genus n torus.

C 3D toric code with boundary

The generation from 2D toric code to 3D toric code with boundary is complicated but direct. We can continue to use a plaquette as the basic structure but consider four different types of cubes. Let us take the eight cubes in Figure 17 as an example. We begin with the red cube and develop it into pink cubes. Orange cubes are the next and the yellow cube completes the model. In the following, we will divide the method into four steps, each step describes one type of cubes.

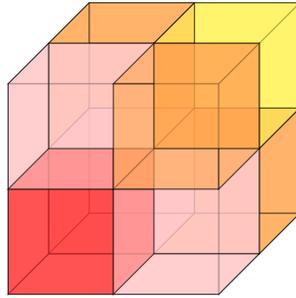


Figure 17: The beginning cube is colored red. The pink, orange and yellow cube represent the cubes connected with one, two or three faces developed.

To develop the qubits in the beginning red cube, we need to develop five rather than six faces as the cube is a closed surface with one redundant face. As shown in Figure 18, we develop a face first and choose the four qubits on the opposite face to repeat the basic structure. After that, considering the pink cube shares a face with developed cube, we only need to develop four more faces as the second cube is also a closed surface. We choose the four qubits on the face opposite to the developed cube to repeat the basic structure.

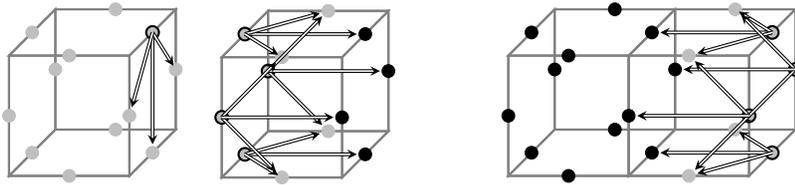


Figure 18: The left two cubes describe the first step to develop the red cube. The right cubes describes the second step to develop the pink cube.

Similarly, we need to develop three faces for the orange cubes and two faces for the yellow cube as shown in Figure 19. The four steps complete the procedure to simulate the ground state of toric code on the eight cubes lattices. And we

are able to develop any size cubes with boundary using the method described above.

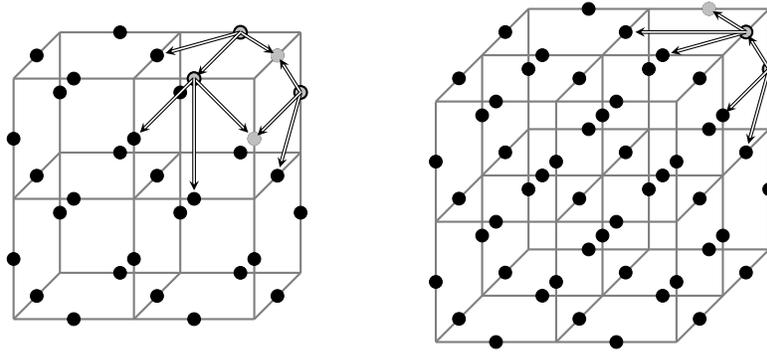


Figure 19: The left figure describes the step of orange cubes, and we need to develop the face in front first. The right figure describes the final step to develop the yellow cube, and we need to develop the face above first.