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Framed cord algebra invariant of knots in $S^1 \times S^2$

Shawn X. Cui

Department of Mathematics,
University of California,
Santa Barbara, CA 93106, USA
xingshan@math.ucsb.edu

Zhenghan Wang

Department of Mathematics,
University of California,
Santa Barbara, CA 93106, USA

Microsoft Research, Station Q,
University of California,
Santa Barbara, CA 93106, USA
zhenghwa@microsoft.com

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ABSTRACT

We generalize Ng's two-variable algebraic/combinatorial zeroth framed knot contact homology for framed oriented knots in S^3 to knots in $S^1 \times S^2$, and prove that the resulting knot invariant is the same as the framed cord algebra of knots. Actually, our cord algebra has an extra variable, which potentially corresponds to the third variable in Ng's three-variable knot contact homology. Our main tool is Lin's generalization of the Markov theorem for braids in S^3 to braids in $S^1 \times S^2$. We conjecture that our framed cord algebras are always finitely generated for non-local knots.

Keywords: Knot; braid group; knot contact homology; cord algebra.

Mathematics Subject Classification 2010: 57M27, 57R17

1. Introduction

The dream of finding new higher categorical quantum invariants of smooth 4-manifolds that can distinguish smooth structures beyond Donaldson/Seiberg-Witten/Heegaard-Floer theory is largely unrealized, despite the spectacular success for new invariants in 3-dimensions and recent progress in higher category theory. A potentially new quantum invariant would be to promote the relative knot contact homology of knots in S^3 in [9] to a $(3+1)$-TQFT-type theory (presumably the zeroth part of the BRST cohomology of a topological string theory). One lesson from $(2+1)$-dimensions is the emergence of powerful diagrammatical techniques as exemplified by the Kauffman bracket definition of the Jones polynomial, and the subsequently elementary formulation of Turaev–Viro and Reshetikhin–Turaev $(2+1)$-TQFTs. We see a striking parallel between the cord algebra invariant and the Jones polynomial.

In [9], the zeroth part of the relative knot contact homology in S^3 is interpreted using cords and skein relations—the main ingredients of diagrammatical techniques in $(2+1)$-dimensions, analogous to the reformulation of the Jones polynomial of knots from von Neumann algebra using knot diagrams and the Kauffman bracket. Taking the elementary framed cord algebra invariant of knots in general 3-manifolds M as the main object of interest, we will follow the diagrammatical approach to constructing $(2+1)$-TQFTs such as the Turaev–Viro and Reshetikhin–Turaev TQFTs. As a first step, we generalize Ng's two-variable combinatorial/algebraic zeroth framed knot contact homology for framed oriented knots in S^3 to knots in $S^1 \times S^2$, and prove that the resulting knot invariant is the same as the framed cord algebra of knots. Actually, our cord algebra has an extra variable, which potentially corresponds to the third variable in Ng's three-variable knot contact homology [10].

It is conjectured in [8] that the cord algebra invariant of knots in a general 3-manifold M is the zeroth relative knot contact homology. We do not prove this conjecture and will not use any knot contact homology theory. Instead we provide an algebraic version of this conjectured zeroth knot contact homology for knots in $S^1 \times S^2$ following [9] and regard our algebraic definition of the cord algebra as an effective method to calculate the topologically defined cord algebra invariant of knots. Our long term goal is to understand the higher categories underlying this algebraic formulation with an eye toward to a diagram construction of a $(3+1)$-TQFT-type theory.

A second reason for our interest in the framed cord algebra invariant of knots is the conjectured relation between the augmentation polynomial and the Homfly polynomial of knots. A well-known question since the discovery of the Jones polynomial is how to place the Jones polynomial within classical topology (since knots are determined by their complements, so any knot invariant is determined by the homeomorphism type of the knot complement). The cord algebra of a knot is basically within classical topology, so the establishment of the conjectured relation between the augmentation polynomial and the Homfly polynomial is one answer to an old question.

To generalize the algebraic zeroth knot contact homology in [9] from S^3 to $S^1 \times S^2$, we use Lin's generalization of the Markov theorem for braids in S^3 to braids in $S^1 \times S^2$ [6] developed for defining a Jones polynomial of knots in $S^1 \times S^2$.[a]

The rest of the paper is organized as follows. In Sec. 2.1, we introduce the Markov theorem for knots in $S^1 \times S^2$, which are represented by the closure of

[a] This generalization, eventually rendered unnecessary for the intended application by Witten's work, finds a similar application in our work. We dedicate our work to X.-S. Lin—an important vanguard in quantum knot theory.

elements in $\mathcal{C}_n$, the Artin group with Dynkin diagram B_n. In Sec. 2.2, we give several actions of $\mathcal{C}_n$ on free algebras. We interpret these actions both algebraically and topologically. These actions will be the key ingredients to define the invariant HC_0 in Sec. 3.1. In Secs. 3.2–3.4, we compute some specific examples, demonstrate some useful propositions, and prove the invariance of HC_0 under Markov moves, respectively. Sections 4.1–4.4 are devoted to prove several properties of the HC_0 invariant. We study two special classes of knots in $S^1 \times S^2$, torus knots and local knots. Moreover, we derive a family of invariants, called augmentations, from HC_0. Finally, in Sec. 5, we prove that the HC_0 invariant has a nice topological interpretation as the framed cord algebra defined in [9].

The first author also created a Mathematica package for computer calculations of the HC_0 invariant and augmentation numbers. The program can be found at [3] and is partly motivated by Ng's computer package, which was used to compute various invariants derived from knot contact homology for knots in S^3. To run the program, one needs to install the non-commutative algebra package NCAlgebra/NCGB [5].

2. Markov Moves and Actions of $\mathcal{C}_n$ on Free Algebras

First we provide some background materials. Links and knots in this paper are always framed and oriented.

2.1. *Markov Moves in $S^1 \times S^2$*

In this subsection, we describe a theorem on Markov moves for links in $S^1 \times S^2$. See [6] for a more detailed discussion.

The classical braid group with n strands, $\mathcal{B}_n$, is defined by the presentation $\langle \sigma_1, \ldots, \sigma_{n-1} \,|\, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2\rangle$. It is the Artin group with Dynkin diagram type A_{n-1}, and can also be viewed as the braid group on the 2-disk $D^2 \subset \mathbb{R}^2$.

Any link in S^3 can be represented as the closure of some braid in the classical braid group. The Markov theorem states that two braids B, B' give rise to the same link if and only if B' can be obtained from B by a finite sequence of the following operations or their inverses:

(1) change $B \in \mathcal{B}_n$ to one of its conjugates in $\mathcal{B}_n$;
(2) change $B \in \mathcal{B}_n$ to $B\sigma_n^{\pm 1} \in \mathcal{B}_{n+1}$.

The Markov theorem for links in S^3 is generalized to links in $S^1 \times S^2$ in [6] as follows.

Let $\mathcal{C}_n$ be the Artin group corresponding to the Dynkin diagram B_n generated by $\alpha_0, \ldots, \alpha_{n-1}$, with the following generating relations:

(1) $\alpha_i\alpha_j = \alpha_j\alpha_i, |i-j| \geq 2$

(2) $\alpha_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i \alpha_{i+1}, i \geq 1$
(3) $\alpha_0 \alpha_1 \alpha_0 \alpha_1 = \alpha_1 \alpha_0 \alpha_1 \alpha_0$.

A direct consequence of the presentation of $\mathcal{C}_n$ is that there are natural inclusions $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_n \subset \cdots$. Denote by ϵ^- these natural inclusions.

It is shown in [2] that $\mathcal{C}_n$ is isomorphic to the braid group on the annulus $[0,1] \times S^1$, or the 1-punctured disk. Specifically, the isomorphism is illustrated in Fig. 1.

Simply treating $\{\text{puncture}\} \times [0,1]$ as the first strand of the new braid, we can regard a braid on the 1-punctured disk as a braid on the disk. Thus we have an embedding of $\mathcal{C}_n$ into $\mathcal{B}_{n+1}$. Denote the generators of $\mathcal{B}_{n+1}$ by $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$. Then the embedding from $\mathcal{C}_n$ to $\mathcal{B}_{n+1}$ is given by the following map:

$$\mathcal{C}_n \to \mathcal{B}_{n+1}, \quad \alpha_0 \mapsto \sigma_0^2, \quad \alpha_i \mapsto \sigma_i, i \geq 1.$$

We will identify $\mathcal{C}_n$ with its image in B_{n+1}, which is the subgroup consisting of the braids that fix the first puncture.

The correspondence between braids on the annulus and links in $S^1 \times S^2$ is obtained via open book decompositions.

Consider the standard open book decomposition of S^3 with an unknot J as the binding. Let K be another unknot which is a closed braid with respect to the braid axis J. Then

$$M = \overline{S^3 \backslash (J \times D^2 \cup K \times D^2)}$$

is a fibration over S^1 whose fiber is an annulus $[0,1] \times S^1$. $S^1 \times S^2$ is obtained by a 0-Dehn surgery along K. Thus $S^1 \times S^2 = M \sqcup_f D^2 \times S^1$, where f is the gluing homeomorphism which maps the meridian of the solid torus to $K \times z_0, z_0 \in \partial D^2$. Let K^* be the image of $0 \times S^1$ under f in $S^1 \times S^2$, where $0 \times S^1$ is the core of

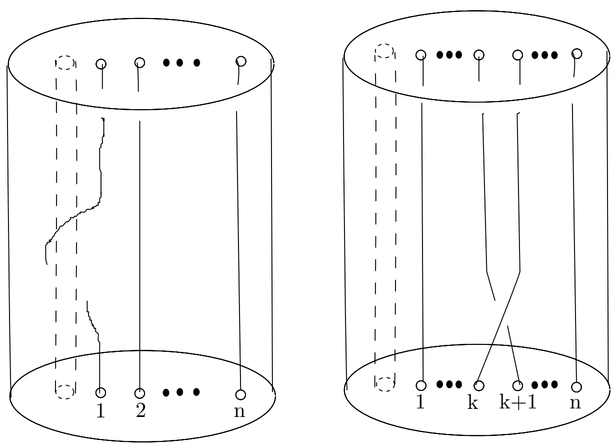


Fig. 1. α_0 and $\alpha_k, k \geq 1$.

the solid torus. We call K^* the dual knot of K. Then the fibration on M extends to an open book decomposition on $S^1 \times S^2$ with the binding $J \cup K^*$. Note that $S^1 \times S^2 \backslash (J \cup K^*)$ is homeomorphic to the product of the annulus with S^1. Any link in $S^1 \times S^2$ can be isotoped into $S^1 \times S^2 \backslash (J \cup K^*)$ transversal to each page, and thus becomes a braid on the annulus.

To state the Markov theorem, we need one more lemma.

Define a map $\epsilon^+ : \mathcal{C}_n \to \mathcal{C}_{n+1}$,

$$\epsilon^+(\alpha_i) = \begin{cases} \alpha_1\alpha_0\alpha_1 & i = 0, \\ \alpha_{i+1} & i \geq 1. \end{cases} \tag{2.1}$$

The map ϵ^+ has a nice geometrical interpretation if we view $\mathcal{C}_n$ as the braid group on the annulus. The map simply inserts a straight strand right next to the line $\{\text{puncture}\} \times [0, 1]$. See Fig. 2.

Note that the newly inserted line will be labeled by 1, and the other strands' labels will be shifted up by 1.

Lemma 2.1 ([6]). *The map ϵ^+ is an injective group homomorphism.*

Proof. From the geometrical interpretation of the map, it should be clear that it is an injective group homomorphism. For a rigorous algebraic proof, see [6]. □

Remark 2.2. Now there are two embeddings of $\mathcal{C}_n$ into $\mathcal{C}_{n+1}$, namely the natural inclusion ϵ^- and the map ϵ^+. From the geometric point of view, ϵ^- is to place a strand on the far right of the braid, while ϵ^+ is to insert a strand right next to the line $\{\text{puncture}\} \times [0, 1]$.

Here is the statement of the Markov Theorem for links in $S^1 \times S^2$.

Theorem 2.3 ([6]). *The closures of two braids $\beta, \beta' \in \bigcup_{n=1}^{\infty} \mathcal{C}_n$ give the same link in $S^1 \times S^2$ if and only if there is a finite sequence of braids, $\beta = \beta_0, \beta_1, \ldots, \beta_k = \beta'$,*

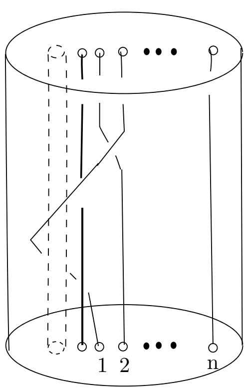


Fig. 2. $\epsilon^+(\alpha_1\alpha_0)$.

such that β_{i+1} can be obtained from β_i by one of the following operations or their inverses:

(1) *change $\beta_i \in \mathcal{C}_n$ to one of its conjugates in $\mathcal{C}_n$*;
(2) *change $\beta_i \in \mathcal{C}_n$ to $\epsilon^-(\beta_i)\alpha_n^{\pm} \in \mathcal{C}_{n+1}$*;
(3) *change $\beta_i \in \mathcal{C}_n$ to $\epsilon^+(\beta_i)\alpha_1^{\pm} \in \mathcal{C}_{n+1}$*.

Remark 2.4. Given a braid $\beta \in \mathcal{C}_n$, we can obtain the knot in $S^1 \times S^2$ represented by β as follows. Take a punctured disk $D' = D \backslash B_\epsilon(0)$, and let $X = D' \times [0,1]$. Draw the diagram of β inside X. Then $S^1 \times S^2$ is obtained by identifying the top and the bottom punctured disk and then gluing a solid torus to each torus boundary component. The gluing maps are given by sending the meridian of each solid torus to $z_0 \times S^1$ and $z_1 \times S^1$, respectively, for some z_0 on the boundary of the puncture and z_1 on the outer boundary of D'. The knot represented by β is the image of the braid diagram in $S^1 \times S^2$. See Fig. 3 for $\beta = \alpha_0\alpha_1$.

2.2. *Actions of $\mathcal{C}_n$ on free algebras*

Throughout the paper, R denotes the commutative ring $\mathbb{Z}[\lambda^{\pm}, \mu^{\pm}, \Gamma^{\pm}]$. Also, the multiplication sign in an algebra is denoted by the symbol $\otimes$ or simply omitted, while the symbol $*$ means concatenation of two curves or some analogous operation in an algebra which will be introduced in Definition 2.13. We will always omit the multiplication sign when writing the product of two elements in a group. We define several free non-commutative algebras over the ring R as follows.

$$\mathcal{A}_n^+ := R\langle a_{ij}^x, 0 \le i,j \le n, x \in \mathbb{Z}\rangle / \langle a_{ii}^0 - (1+\mu)\Gamma, 0 \le i \le n\rangle,$$

$$\mathcal{A}_n^- := R\langle a_{ij}^x, 1 \le i,j \le n+1, x \in \mathbb{Z}\rangle / \langle a_{ii}^0 - (1+\mu)\Gamma, 1 \le i \le n+1\rangle,$$

$$\mathcal{A}_n := R\langle a_{ij}^x, 1 \le i,j \le n, x \in \mathbb{Z}\rangle / \langle a_{ii}^0 - (1+\mu)\Gamma, 1 \le i \le n\rangle.$$

The algebra $\mathcal{A}_n$ can be embedded into $\mathcal{A}_n^+$ and $\mathcal{A}_n^-$ in the most natural way. We will always identity $\mathcal{A}_n$ with its images in $\mathcal{A}_n^+$ and $\mathcal{A}_n^-$.

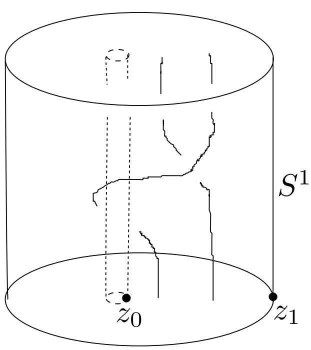


Fig. 3. The closure of β in $S^1 \times S^2$.

Now we introduce an action of $\mathcal{C}_n$ on $\mathcal{A}_n$, and extend the action to the larger algebras $\mathcal{A}_n^+, \mathcal{A}_n^-$. The action is first presented algebraically and then will be given a topological interpretation.

2.2.1. *Algebraic interpretation of the actions*

Recall that the generators $\mathcal{C}_n$ are denoted by $\alpha_0, \ldots, \alpha_{n-1}$, which satisfy the relation given in Sec. 2.1. We define a group morphism $\Phi : \mathcal{C}_n \to \mathring{A}ut(\mathcal{A}_n)$ as follows.

For $1 \leq k \leq n-1$, $1 \leq i, j \leq n$,

$$\Phi(\alpha_k)(a_{ij}^x) = \begin{cases} -a_{k+1,j}^x + \frac{1}{\Gamma\mu} a_{k+1,k}^0 a_{k,j}^x & i = k, j \neq k, k+1, \\ -a_{k+1,k}^x + \frac{1}{\Gamma\mu} a_{k+1,k}^0 a_{k,k}^x & i = k, j = k+1, \\ a_{k+1,k+1}^x - \frac{1}{\Gamma} a_{k+1,k}^x a_{k,k+1}^0 & \\ -\frac{1}{\Gamma\mu} a_{k+1,k}^0 a_{k,k+1}^x + \frac{1}{\Gamma^2\mu} a_{k+1,k}^0 a_{k,k}^x a_{k,k+1}^0 & i = k, j = k, \\ a_{k,j}^x & i = k+1, j \neq k, k+1, \\ -a_{k,k+1}^x + \frac{1}{\Gamma} a_{k,k}^x a_{k,k+1}^0 & i = k+1, j = k, \\ a_{k,k}^x & i = k+1, j = k+1, \\ -a_{i,k+1}^x + \frac{1}{\Gamma} a_{i,k}^x a_{k,k+1}^0 & i \neq k, k+1, j = k, \\ a_{i,k}^x & i \neq k, k+1, j = k+1, \\ a_{i,j}^x & i \neq k, k+1, j \neq k, k+1, \end{cases} \tag{2.2}$$

$$\Phi(\alpha_0)(a_{ij}^x) = \begin{cases} a_{1,1}^x & i = 1, j = 1, \\ -\mu a_{1,j}^{x-1} + \frac{1}{\Gamma} a_{1,1}^x a_{1,j}^{-1} & i = 1, j \geq 2, \\ \frac{1}{\mu}\left(-a_{i,1}^{x+1} + \frac{1}{\Gamma} a_{i,1}^1 a_{1,1}^x\right) & i \geq 2, j = 1, \\ a_{i,j}^x - \frac{1}{\Gamma\mu} a_{i,1}^{x+1} a_{1,j}^{-1} & \\ -\frac{1}{\Gamma} a_{i,1}^1 a_{1,j}^{x-1} + \frac{1}{\Gamma^2\mu} a_{i,1}^1 a_{1,1}^x a_{1,j}^{-1} & i \geq 2, j \geq 2. \end{cases} \tag{2.3}$$

It is direct, though tedious, to check that Φ is well-defined, i.e. $\Phi(\alpha_i)$ satisfies the braid relations that define $\mathcal{C}_n$. Alternatively, in Sec. 2.2.2, we will describe the mapping class action of a braid in $\mathcal{C}_n$ on $\tilde{\mathcal{A}}_n$, a quotient of an algebra generated by paths in a punctured disk. Then $\tilde{\mathcal{A}}_n$ will be shown to be generated by simple paths and isomorphic to $\mathcal{A}_n$. Upon identifying these simple paths with the elements a_{ij}^x, the map determined by Eqs. (2.2) and (2.3) is seen to agree with this action. See Theorems 2.7 and 2.8.

We extend the action of $\mathcal{C}_n$ to the algebra $\mathcal{A}_n^+$ by furthermore defining the action on $a_{0j}^x, a_{i0}^x, 0 \leq i, j \leq n$. This extended action is denoted by Φ^+.

$$\Phi^+(\alpha_0)(a_{ij}^x) = \begin{cases} a_{0,0}^x & i=0, j=0, \\ \dfrac{1}{\mu} a_{0,1}^{x+1} & i=0, j=1, \\ -a_{0,j}^x + \dfrac{1}{\Gamma\mu} a_{0,1}^{x+1} a_{1,j}^{-1} & i=0, j\geq 2, \\ \mu a_{1,0}^{x-1} & i=1, j=0, \\ -a_{i,0}^x + \dfrac{1}{\Gamma} a_{i,1}^1 a_{1,0}^{x-1} & i\geq 2, j=0. \end{cases} \tag{2.4}$$

For $1 \leq k \leq n-1$, $\Phi^+(\alpha_k)(a_{ij}^x)$ are given by the same equation as Eq. (2.2), except that now i, j are allowed to be zero when they are not k or $k+1$.

Similarly, the extended action of $\mathcal{C}_n$ on $\mathcal{A}_n^-$ is defined by Eqs. (2.2) and (2.3) except that the range of i, j now is from 1 to $n+1$. We denote this action by Φ^-.

Again, one can check directly Φ^+, Φ^- are both well-defined. Alternatively, see Theorem 2.10.

A few remarks are in order.

Remark 2.5. (1) For a braid $\beta \in C_n$, we will write $\Phi_\beta, \Phi_\beta^+, \Phi_\beta^-$ for $\Phi(\beta), \Phi^+(\beta)$, $\Phi^-(\beta)$, respectively, in subsequent sections.

(2) From the definitions of the actions mentioned above, we can see that $\Phi_\beta = (\Phi_\beta^+)_{|\mathcal{A}_n} = (\Phi_\beta^-)_{|\mathcal{A}_n}$, and that $\Phi_\beta^- = \Phi_{\epsilon^-(\beta)}$ if we identify $\mathcal{A}_n^-$ with $\mathcal{A}_{n+1}$ in the obvious way.

(3) Denote by $\mathcal{B}_n$ the subgroup of $\mathcal{C}_n$ generated by $\{\alpha_1, \ldots, \alpha_{n-1}\}$. Then $\mathcal{B}_n$ is isomorphic to the classical braid group on n strands. In Eq. (2.2), if we set $\Gamma = -1, \mu = 1$, and $x = 0$, then $\Phi_{|\mathcal{B}_n}$ acting on $\mathbb{Z}\langle a_{ij}^0 \rangle$ is exactly the braid group action given in [7]. So our braid group action is a generalization of the action in [7].

The above actions will be less mysterious after we give a topological interpretation in the following subsection.

2.2.2. *Topological interpretation of the actions*

Let D be the unit disk in the complex plane centered at the origin, D_n be the punctured disk with $n+1$ punctures labeled, from left to right, by $p, p_1, \ldots, p_n$ and let $q_i = p_i - \epsilon, 1 \leq i \leq n$, $\epsilon > 0$ be n points in D_n each close to the corresponding puncture. See Fig. 4.

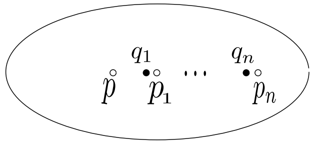


Fig. 4. D_n.

Let $Q_n = \{q_i, 1 \leq i \leq n\}$ and let $\mathcal{Q}_n = \{\gamma : [0,1] \to D_n \,|\, \gamma$ is continuous, $\gamma(0), \gamma(1) \in Q_n\} \sim$ be the set of equivalence classes of curves in D_n with end points belonging to Q_n. Here the equivalence relation $\sim$ means homotopy relative to end points. So the curves are not allowed to pass through any of the punctures and their end points are fixed during the homotopy.

Let $\tilde{\mathcal{A}}_n$ be the free non-commutative algebra over R generated by elements of $\mathcal{Q}_n$ modulo the "skein" relations shown in Fig. 5. Note that $\otimes$ in Fig. 5 means the multiplication in $\tilde{\mathcal{A}}_n$. And the second relation, as well as other similar relations in the context, depicts some local neighborhood of the diagrams outside of which they all agree.

For $1 \leq i, j \leq n, x \in \mathbb{Z}$, let γ_{ij}^x and γ_i be the curves shown in Fig. 6, namely γ_{ij}^x starts from q_i, winds around p counter clock-wise x times if $x \geq 0$, or clock-wise $-x$ times if $x < 0$, and finally goes through the upper half disk to end at q_j. The curve γ_i starts and ends at q_i and winds around p_i counter clock-wise once.

It should be noted that the relations shown in Fig. 7 can be derived from the ones in Fig. 5. And the second relation in Fig. 7 is equivalent to the property that if $\gamma, \gamma' \in \mathcal{Q}_n$ such that $\gamma(0) = q_i$ and $\gamma'(1) = q_i$, then $\gamma_i * \gamma = \mu\gamma, \gamma' * \bar{\gamma}_i = \mu^{-1}\gamma'$,

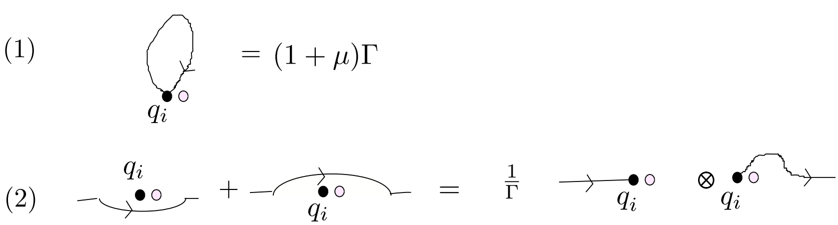


Fig. 5. Skein relations $\tilde{\mathcal{A}}_n$.

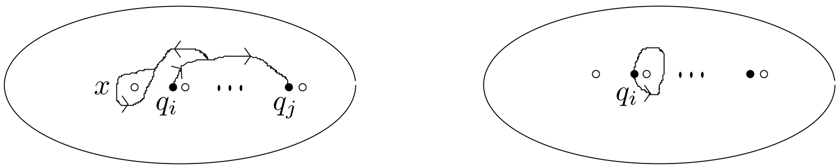


Fig. 6. γ_{ij}^x and γ_i.

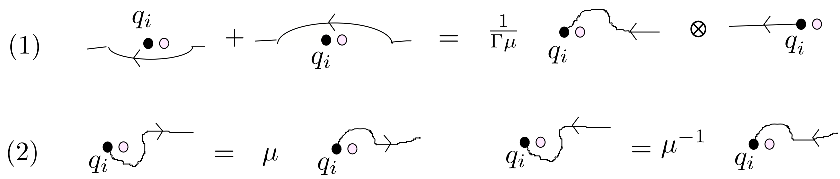


Fig. 7. Derived skein relations.

where $*$ means concatenation of two curves, and $\bar{\gamma}_i$ is the curve γ_i with reversed direction.

We will show below that there is an isomorphism between $\tilde{\mathcal{A}}_n$ and $\mathcal{A}_n$ and that γ_{ij}^x is identified with a_{ij}^x under this isomorphism.

Pick a base point on the boundary of the disk D_n. To make it explicit, let us pick some z_0 on the upper half of the boundary as the base point. The fundamental group of D_n is the free group F_{n+1} on $n+1$ generators, which we denote by $e, e_1, \ldots, e_n$, where e_i is the loop that winds around p_i counter clock-wise once and e is the loop that winds around p counter clock-wise once. See Fig. 8.

For each $1 \leq i \leq n$, let δ_i be the straight line from z_0 to q_i, and $\bar{\delta}_i$ be the same line with reversed direction. For any curve $\gamma \in \mathcal{Q}_n$ with $\gamma(0) = q_i, \gamma(1) = q_j$, let $\tilde{\gamma} = \delta_i * \gamma * \bar{\delta}_j$, then $\tilde{\gamma}$ becomes an element in $\pi_1(D_n, z_0) = F_{n+1}$.

For $\tilde{\gamma} \in \pi_1(D_n, z_0)$, let $l(\tilde{\gamma})$ be the minimum number of occurrences of $e_i^{\pm 1}, 1 \leq i \leq n$ in the words representing $\tilde{\gamma}$. So we do not count the occurrences of e in computing $l(\tilde{\gamma})$.

The following proposition shows that by repeated applications of the "skein" relations in Fig. 5, any element of $\tilde{\mathcal{A}}_n$ can be reduced to a (non-commutative) polynomial in γ_{ij}^x's.

Proposition 2.6. *The algebra $\tilde{\mathcal{A}}_n$ is generated by $\{\gamma_{ij}^x : 1 \leq i, j \leq n, i \neq j, x \in \mathbb{Z}\}$.*

Proof. Since $\tilde{\mathcal{A}}_n$ is generated by elements of $\mathcal{Q}_n$, it suffices to show that each element of $\mathcal{Q}_n$ can be written as a polynomial of γ_{ij}^x's. We prove this statement by induction on $l(\tilde{\gamma})$ for $\gamma \in \mathcal{Q}_n$.

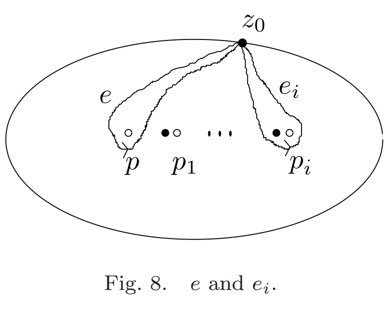


Fig. 8. e and e_i.

If $l(\tilde{\gamma}) = 0$, then $\tilde{\gamma} = e^x$ for some $x \in \mathbb{Z}$, and so γ is equal to some γ_{ij}^x.

Assume the statement is true for all γ with $l(\tilde{\gamma}) \leq m-1, m \geq 1$. Let $\gamma \in \mathcal{Q}_n$ be any element such that $l(\tilde{\gamma}) = m$. Choose a word w representing $\tilde{\gamma}$ such that the number of occurrences of $e_i^{\pm 1}$'s in w is m. Then there exists some k, $1 \leq k \leq n$, such that $w = w_0 e_k w_1$ or $w = w_0 e_k^{-1} w_1$, where w_0 and w_1 are sub-words (possibly empty).

If $w = w_0 e_k w_1$, apply the second relation in Fig. 5 to γ around the puncture p_k with γ being the first term on the left-hand side. Then there exist $\gamma_0, \gamma_1, \gamma' \in \mathcal{Q}_n$, such that $\gamma = -\gamma' + \frac{1}{\Gamma}\gamma_0\gamma_1$, and that $\tilde{\gamma_0} = w_0, \tilde{\gamma_1} = w_1, \tilde{\gamma'} = w_0 w_1$. Since $l(w_0), l(w_1), l(w_0 w_1)$ are all less than m, by induction, γ_0, γ_1 and γ' are polynomials of γ_{ij}^x's, and thus γ is also a polynomial of γ_{ij}^x's.

The case $w = w_0 e_k^{-1} w_1$ can be proved analogously by referring to the first relation in Fig. 7. $\square$

We proceed to prove $\{\gamma_{ij}^x : 1 \leq i, j \leq n, i \neq j, x \in \mathbb{Z}\}$ are actually free generators of $\tilde{\mathcal{A}}_n$.

Define an intermediate non-commutative algebra $\mathcal{B} = R\langle e^{\pm 1}, y_1, y_2, \ldots, y_n\rangle/\mathcal{I}$, where $\mathcal{I}$ is the two-sided ideal generated by $ee^{-1} - 1$, $e^{-1}e - 1$ and $y_i^2 - \Gamma(1+\mu)y_i, 1 \leq i \leq n$. We define a multiplicative map from F_{n+1} to $\mathcal{B}$ as follows.

$$\tau : F_{n+1} \to \mathcal{B},$$
$$\tau(w) = \begin{cases} \dfrac{1}{\Gamma} y_i - 1 & w = e_i, 1 \leq i \leq n, \\ \dfrac{1}{\Gamma\mu} y_i - 1 & w = e_i^{-1}, 1 \leq i \leq n, \\ e^{\pm 1} & w = e^{\pm 1}, \\ 1 & w = 1. \end{cases} \tag{2.5}$$

It follows immediately that $\tau(e)\tau(e^{-1}) = \tau(e_i)\tau(e_i^{-1}) = 1 = \tau(1)$ in $\mathcal{B}$. Therefore, we can extend the action of τ uniquely to arbitrary words to get a well-defined multiplicative map on F_{n+1}. Actually τ extends to an algebra morphism from the group ring $R[F_{n+1}]$ to $\mathcal{B}$.

For $1 \leq i, j \leq n$, we define an R-linear map $\alpha_{ij} : R\langle e^{\pm 1}, y_1, y_2, \ldots, y_n\rangle \to \mathcal{A}_n$,

$$\alpha_{ij}(e^{i_1} y_{j_1} e^{i_2} y_{j_2} \cdots e^{i_k} y_{j_k} e^{i_{k+1}}) := a_{i,j_1}^{i_1} a_{j_1,j_2}^{i_2} \cdots a_{j_{k-1},j_k}^{i_k} a_{j_k,j}^{i_{k+1}}$$

One can verify that α_{ij} factors through $\mathcal{I}$ using the fact that $a_{ii}^0 = (1+\mu)\Gamma$. Thus α_{ij} induces a map from $\mathcal{B}$ to $\mathcal{A}_n$, which is still denoted by α_{ij}.

Define a map $\psi : \mathcal{Q}_n \to \mathcal{A}_n$ by $\psi(\gamma) := \alpha_{ij}\tau(\tilde{\gamma})$, where γ is an element of $\mathcal{Q}_n$ such that $\gamma(0) = q_i, \gamma(1) = q_j$, and $\tilde{\gamma} = \delta_i * \gamma * \bar{\delta_j}$. We extend ψ multiplicatively to the free R-algebra generated by elements of $\mathcal{Q}_n$. In Theorem 2.7, it will be proved that this extended map factors through the "skein relations" shown in Fig. 5, thus it induces a map, still denoted by ψ, from $\tilde{\mathcal{A}}_n$ to $\mathcal{A}_n$.

Theorem 2.7. *The map ψ introduced above is a well-defined algebra isomorphism from $\tilde{\mathcal{A}}_n$ to $\mathcal{A}_n$ sending γ_{ij}^x to a_{ij}^x.*

Proof. Clearly, $\psi(\gamma)$ is independent of the choice of representatives of γ in its equivalence class.

We first show ψ factors through the "skein" relations in Fig. 5.

It is easily seen that $\psi(\gamma_{ij}^x) = a_{ij}^x$. In particular, $\psi(\gamma_{ii}^0) = a_{ii}^0 = (1+\mu)\Gamma$, so the first "skein" relation is preserved ψ.

Let C_1, C_2 denote the two curves passing above and below p_k, respectively, in the definition of the second "skein" relation in Fig. 5. They have the same initial and end points, say q_i, q_j. Let C_3, C_4 be the curves which ends at q_k and starts at q_k, respectively. So C_3 starts from q_i and C_4 ends at q_j. Let w_3, w_4 be the words in F_{n+1} which represent $\tilde{C}_3, \tilde{C}_4$, then the words which represent $\tilde{C}_1, \tilde{C}_2$ are $w_3w_4, w_3e_kw_4$.

Thus, $\psi(C_1) + \psi(C_2) = \alpha_{ij}(\tau(w_3)\tau(w_4)) + \alpha_{ij}(\tau(w_3)(\frac{1}{\Gamma}y_k - 1)\tau(w_4)) = \frac{1}{\Gamma}\alpha_{ij}(\tau(w_3)y_k\tau(w_4)) = \frac{1}{\Gamma}\alpha_{ik}(\tau(w_3))\alpha_{kj}(\tau(w_4)) = \frac{1}{\Gamma}\psi(C_3)\psi(C_4)$, which says ψ preserves the second "skein" relation.

The above arguments show that ψ is a well-defined algebra morphism. Define the inverse map $\psi' : \mathcal{A}_n \to \tilde{\mathcal{A}}_n$ by sending each a_{ij}^x to γ_{ij}^x. Noting that γ_{ij}^x are generators of $\tilde{\mathcal{A}}_n$ by Proposition 2.6, we have $\psi\psi' = \mathrm{Id}$ and $\psi'\psi = \mathrm{Id}$. Therefore, ψ is an algebra isomorphism. □

Now we describe a natural action of $\mathcal{C}_n$ on $\tilde{\mathcal{A}}_n$.

Recall that the group of isotopy classes of homeomorphisms of D_n with boundary fixed point-wise is the classical braid group on $n+1$ strands $\mathcal{B}_{n+1}$.[b] Here we assume the generators are $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$, where σ_0 is the Dehn twist that switches p with p_1 counter clock-wise and σ_i switches p_i with $p_{i+1}, 1 \leq i \leq n-1$. Also recall that we identified $\mathcal{C}_n$ with the subgroup of $\mathcal{B}_{n+1}$ which consists of the braids that fix the first puncture. See Sec. 2.1 for the explicit embedding. Therefore, the elements of $\mathcal{C}_n$ fix the puncture p and permute $\{p_i, 1 \leq i \leq n\}$. We can furthermore stipulate that the horizontal line segments p_iq_i remain horizontal and of fixed length during the isotopy, so that the elements of $\mathcal{C}_n$ also permute the q_i's. It follows that the elements of $\mathcal{C}_n$ act on $\mathcal{Q}_n$. One can also check that this action actually preserves the "skein" relations. Therefore, we get a natural action $\tilde{\Phi}$ of $\mathcal{C}_n$ on $\tilde{\mathcal{A}}_n$.

Theorem 2.8. *The algebra isomorphism $\psi : \tilde{\mathcal{A}}_n \to \mathcal{A}_n$ preserves the action of $\mathcal{C}_n$, i.e. $\psi\tilde{\Phi}_\beta = \Phi_\beta\psi$, for any $\beta \in \mathcal{C}_n$.*

Proof. It suffices to check for any $\beta = \alpha_k$, $\psi\tilde{\Phi}_\beta = \Phi_\beta\psi$ holds on the generators γ_{ij}^x. We left this as an exercise. □

Remark 2.9. It is worth pointing out that when we want to find the image of some complicated curve in $\tilde{\mathcal{A}}_n$ under ψ, it is usually more efficient to use the "skein"

[b]Note that here D_n has $n+1$ punctures.

relations than using the definition directly. Also, instead of memorizing the action of $\mathcal{C}_n$ on the a^x_{ij}'s, it is much easier to manipulate the "skein" relations and the Dehn twists. This provides us another way to calculate the action of a braid β on a^x_{ij}, namely, first use a sequence of Dehn twists representing β to map γ^x_{ij} to some curve, and then decompose this curve into a polynomial of generators using "skein" relations, finally replace the generators in the polynomial by the corresponding a^x_{ij}'s.

For example, to obtain $\Phi_{\alpha_1^2\alpha_0}(a^0_{12})$, we first compute $\tilde{\Phi}_{\alpha_1^2\alpha_0}(\gamma^0_{12})$ using Dehn twists that represent $\alpha_1^2\alpha_0$. See Fig. 9. Then we decompose the resulting curve using "skein" relations to get the expression

$$\tilde{\Phi}_{\alpha_1^2\alpha_0}(\gamma^0_{12}) = \gamma^{-1}_{12} - \frac{1}{\Gamma}\gamma^{-1}_{11}\gamma^0_{12} + \frac{1}{\Gamma}\gamma^0_{12}\gamma^{-1}_{22} - \frac{1}{\Gamma^2\mu}\gamma^0_{12}\gamma^0_{21}\gamma^{-1}_{12} - \frac{1}{\Gamma^2}\gamma^0_{12}\gamma^{-1}_{21}\gamma^0_{12} + \frac{1}{\Gamma^3\mu}\gamma^0_{12}\gamma^0_{21}\gamma^{-1}_{11}\gamma^0_{12}.$$

Replacing the γ^x_{ij}'s above with a^x_{ij}'s, we obtain the expression for $\Phi_{\alpha_1^2\alpha_0}(a^0_{12})$.

There are analogous topological interpretations of the extended actions of $\mathcal{C}_n$ on $\mathcal{A}^+_n$ and $\mathcal{A}^-_n$. The procedure goes the same as above, and we will only point out what modifications should be made at each step.

First of all, let D^+_n be the punctured disk with punctures $p, p_0, p_1, \ldots, p_n$ arranged from left to right and similarly let D^-_n be the punctured disk with punctures $p, p_1, \ldots, p_n, p_{n+1}$. Also in both cases, still choose the points $q_i = p_i - \epsilon$, for some tiny $\epsilon > 0$. Let $\mathcal{Q}^\pm_n$ be the set of equivalence classes of curves in $D^\pm_n$ which start and end at the q_i's . Define $\tilde{\mathcal{A}}^\pm_n$ to be the R-algebra generated by elements of $\mathcal{Q}^\pm_n$ modulo the "skein" relations in Fig. 10, where $q_\pm = q_0$ in the "+" case and $q_\pm = q_{n+1}$ otherwise.

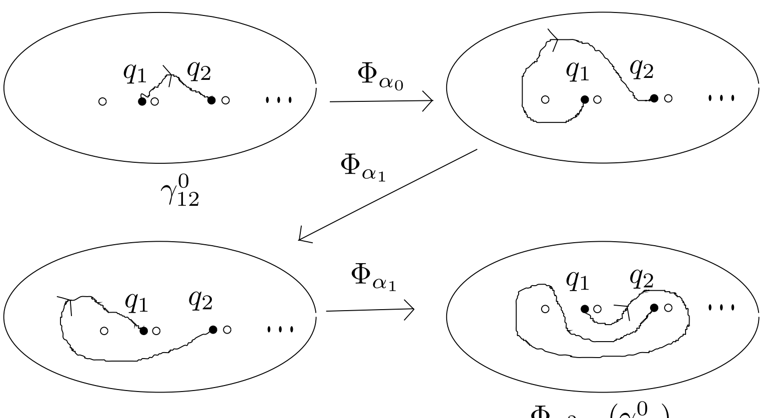


Fig. 9. $\Phi_{\alpha_1^2\alpha_0}(\gamma^0_{12})$.

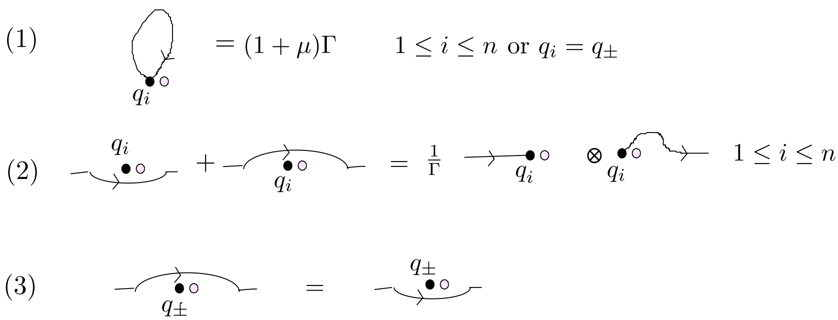


Fig. 10. Skein relations $\tilde{\mathcal{A}}_n^\pm$.

So we added one more relation when defining $\tilde{\mathcal{A}}_n^\pm$, namely, the curves are allowed to pass through the new puncture p_0 (respectively, p_{n+1}).

The fundamental group of $D_n^\pm$ is the free group F_{n+2} generated by $e, e', e_i, 1 \leq i \leq n$, where e' is the generator that corresponds to the new puncture p_0 (respectively, p_{n+1}). We will use the same intermediate algebra $\mathcal{B}$, and the map τ is extended to F_{n+2} by furthermore defining $\tau(e') = 1$.

In the same way as we defined the isomorphism ψ from $\tilde{\mathcal{A}}_n$ to $\mathcal{A}_n$, we can define an isomorphism $\psi^\pm$ from $\tilde{\mathcal{A}}_n^\pm$ to $\mathcal{A}_n^\pm$ which sends γ_{ij}^x to a_{ij}^x.

Next, we extend the action of $\mathcal{C}_n$ to $\tilde{\mathcal{A}}_n^\pm$.

Recall the embedding $\epsilon^+ : \mathcal{C}_n \to \mathcal{C}_{n+1}$ introduced in Sec. 2.1. For notational convenience, we denote the generators of $\mathcal{C}_{n+1}$ by $\alpha_{-1}, \alpha_0, \ldots, \alpha_{n-1}$. Thus the embedding ϵ^+ sends α_0 to $\alpha_0\alpha_{-1}\alpha_0$ and α_i to $\alpha_i, 1 \leq i \leq n-1$. From the geometrical point of view, ϵ^+ simply inserts a strand labeled by p_0 right next to $\{p\} \times [0,1]$. See the first picture in Fig. 11.

Any braid in $\epsilon^+(\mathcal{C}_n)$ fixes p and p_0. Thus, the action of $\mathcal{C}_n$ via the embedding ϵ^+ preserves all the "skein" relations defining $\tilde{\mathcal{A}}_n^+$, and therefore induces an action $\tilde{\Phi}^+$ on $\tilde{\mathcal{A}}_n^+$.

For the action $\tilde{\Phi}^-$ of $\mathcal{C}_n$ on $\tilde{\mathcal{A}}_n^-$, we use the other embedding $\epsilon^- : \mathcal{C}_n \to \mathcal{C}_{n+1}$. Note that here the generators of $\mathcal{C}_{n+1}$ are $\alpha_0, \ldots, \alpha_n$, and $\epsilon^-(\alpha_i) = \alpha_i, 0 \leq i \leq n-1$. The map ϵ^- inserts a strand labeled by p_{n+1} on the right of the braid. See the second picture in Fig. 11.

Note that in Fig. 11 we use i to represent p_i.

Again, since elements of $\epsilon^-(\mathcal{C}_n)$ fix p_{n+1}, they preserve the "skein" relations that define $\tilde{\mathcal{A}}_n^-$. We thus get an induced action $\tilde{\Phi}^-$ of $\mathcal{C}_n$ on $\tilde{\mathcal{A}}_n^-$.

$\tilde{\mathcal{A}}_n$ can obviously be embedded as a subalgebra into $\tilde{\mathcal{A}}_n^\pm$. We have the following theorem which relates the topological interpretations of the actions of $\mathcal{C}_n$ to the algebraic interpretations.

Theorem 2.10. *The maps $\psi^\pm : \tilde{\mathcal{A}}_n^\pm \to \mathcal{A}_n^\pm$ are algebra isomorphisms and commute with the extended actions of $\mathcal{C}_n$, namely, for any $\beta \in \mathcal{C}_n$, $\psi^\pm \tilde{\Phi}_\beta^\pm = \Phi_\beta^\pm \psi^\pm$. Moreover,*

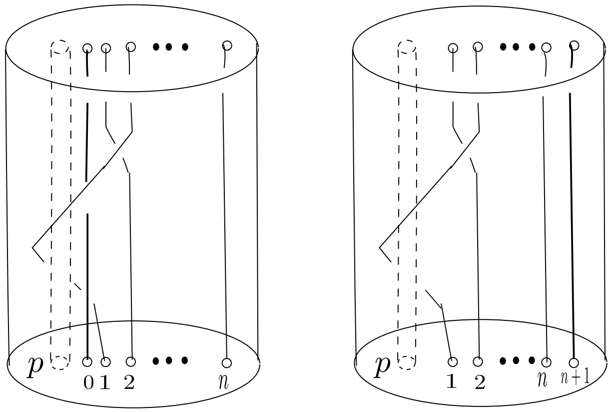


Fig. 11. $\epsilon^+(\alpha_1\alpha_0)$ and $\epsilon^-(\alpha_1\alpha_0)$.

$(\tilde{\Phi}^{\pm}_{\beta})_{|\tilde{\mathcal{A}}_n} = \tilde{\Phi}_\beta$, *and the following diagram commutes*:

$$\begin{tikzcd} \tilde{\mathcal{A}}_n \arrow[r, "\psi"] \arrow[d, hook] & \mathcal{A}_n \arrow[d, hook] \\ \tilde{\mathcal{A}}^{\pm}_n \arrow[r, "\psi^{\pm}"] & \mathcal{A}^{\pm}_n \end{tikzcd} \tag{2.6}$$

And each of the maps in the above diagram preserves the action of $\mathcal{C}_n$.

Proof. Proofs are analogous to that of Theorem 2.7. □

2.2.3. *Properties of the actions*

It is worth noting that the action of $\tilde{\Phi}$ on $\tilde{\mathcal{A}}_n$ and the actions of $\tilde{\Phi}^{\pm}$ on $\tilde{\mathcal{A}}^{\pm}_n$ can also be visualized as follows.

For a braid $\beta \in \mathcal{C}_n$, draw a braid diagram of β inside $D_n \times [0,1]$, such that the intersections of the braid with $D_n \times \{0,1\}$ are exactly the punctures p_i's. Perturb the braid diagram to get a parallel copy of it such that the intersections of the copy with $D_n \times \{0,1\}$ are the q_i's. For any curve $\gamma \subset D_n \times \{0\}$ representing some element in $\tilde{\mathcal{A}}_n$, slide γ along the copy diagram in the complement of the braid diagram until it reaches $D_n \times \{1\}$, then the resulting curve is $\tilde{\Phi}_\beta(\gamma)$.

To visualize $\tilde{\Phi}^{\pm}_{\beta}$, we draw a braid diagram of $\epsilon^{\pm}(\beta)$ inside $D^{\pm}_n \times I$, make a parallel copy of it, and slide any curve along the copy diagram up to $D^{\pm}_n \times \{1\}$.

With the above observations, we have the following simple but important proposition.

Proposition 2.11. *Let* $\beta \in \mathcal{C}_n$ *be a braid, and let* γ_1, γ_2 *be two curves in* $\mathcal{Q}_n$ *(respectively*, $\mathcal{Q}^{\pm}_n$) *such that* $\gamma_1(1) = \gamma_2(0)$, *then* $\tilde{\Phi}_\beta(\gamma_1 * \gamma_2) = \tilde{\Phi}_\beta(\gamma_1) *$

$\tilde{\Phi}_\beta(\gamma_2)$ *(respectively,* $\tilde{\Phi}^\pm_\beta(\gamma_1 * \gamma_2) = \tilde{\Phi}^\pm_\beta(\gamma_1) * \tilde{\Phi}^\pm_\beta(\gamma_2)$*), where* $*$ *again means concatenation of two curves.*

Remark 2.12. For two elements $\gamma_1, \gamma_2 \in \mathcal{Q}_n$ such that $\gamma_1(1) = \gamma_2(0)$, the concatenation $\gamma_1 * \gamma_2$ is different from the product $\gamma_1\gamma_2$ when they are viewed as elements of $\tilde{\mathcal{A}}_n$. For instance, $\gamma^x_{ik} * \gamma^y_{kj} = \gamma^{x+y}_{ij} \neq \gamma^x_{ik}\gamma^y_{kj} \in \tilde{\mathcal{A}}_n$. Note that the product sign in an algebra is either denoted by $\otimes$ or omitted, as stated at the beginning of Sec. 2.2.

We can also define the "$*$" operation on some elements of $\mathcal{A}_n$ (respectively, $\mathcal{A}^\pm_n$).

Definition 2.13. (1) Let $P, Q \in \mathcal{A}_n$(respectively, $\mathcal{A}^\pm_n$) such that $P = \sum_{x\in\mathbb{Z}} \sum_{i=1}^n P^x_i a^x_{ik}, Q = \sum_{y\in\mathbb{Z}} \sum_{j=1}^n a^y_{kj} Q^y_j, P^x_i, Q^y_j \in \mathcal{A}_n$ (respectively, $\mathcal{A}^\pm_n$), then $P * Q \in \mathcal{A}_n$ (respectively, $\mathcal{A}^\pm_n$) is defined to be $\sum_{x,y\in\mathbb{Z}} \sum_{i,j=1}^n P^x_i a^{x+y}_{ij} Q^y_j$.

(2) Two elements $P, Q \in \mathcal{A}_n$ (respectively, $\mathcal{A}^\pm_n$) are called connectable, if they satisfy the condition in the definition above.

The $*$ operation defined on elements of $\mathcal{Q}_n$ and that on elements of $\mathcal{A}_n$ are related by the following proposition.

Proposition 2.14. *Let* $\gamma_1, \gamma_2 \in \mathcal{Q}_n$ (respectively, $\mathcal{Q}^\pm_n$) *such that* $\gamma_1(1) = \gamma_2(0)$, *then* $\psi(\gamma_1 * \gamma_2) = \psi(\gamma_1) * \psi(\gamma_2)$ (respectively, $\psi^\pm(\gamma_1 * \gamma_2) = \psi^\pm(\gamma_1) * \psi^\pm(\gamma_2)$).

Proof. We only prove the case $\gamma_1, \gamma_2 \in \mathcal{Q}_n$. The proof of the other two cases is analogous.

Recall the construction of the isomorphism $\psi : \tilde{\mathcal{A}}_n \to \mathcal{A}_n$ in Sec. 2.2.2. We will also have the notations from Sec. 2.2.2. Assume $\gamma_1(0) = q_i, \gamma_1(1) = \gamma_2(0) = q_k, \gamma_2(1) = q_j$, and let $\tilde{\gamma_1} = \delta_i * \gamma_1 * \bar{\delta_k}$, $\tilde{\gamma_2} = \delta_k * \gamma_2 * \bar{\delta_j}$. Then $\widetilde{\gamma_1 * \gamma_2} = \tilde{\gamma_1}\tilde{\gamma_2}$, where $\tilde{\gamma_1}\tilde{\gamma_2}$ means the multiplication of $\tilde{\gamma_1}$ with $\tilde{\gamma_2}$ in the fundamental group of the punctured disk (but not the concatenation of the two curves). Thus $\psi(\gamma_1 * \gamma_2) = \alpha_{ij}\tau(\tilde{\gamma_1}\tilde{\gamma_2}) = \alpha_{ij}(\tau(\tilde{\gamma_1})\tau(\tilde{\gamma_2}))$, since τ is multiplicative.

If M, N are two monomials in $\mathcal{B}$, then one can get the fact from the definition of α_{ij} that $\alpha_{ik}(M)$ and $\alpha_{kj}(N)$ are connectable and that $\alpha_{ij}(MN) = \alpha_{ik}(M) * \alpha_{kj}(N)$. Extending linearly, this equality holds for M, N two polynomials in $\mathcal{B}$.

Therefore, $\psi(\gamma_1 * \gamma_2) = \alpha_{ij}(\tau(\tilde{\gamma_1})\tau(\tilde{\gamma_2})) = \alpha_{ik}(\tau(\tilde{\gamma_1})) * \alpha_{kj}(\tau(\tilde{\gamma_2})) = \psi(\gamma_1) * \psi(\gamma_2)$. ◻

Proposition 2.15. *If* $P, Q \in \mathcal{A}_n$ (respectively, $\mathcal{A}^\pm_n$) *are connectable, then for any* $\beta \in \mathcal{C}_n$, $\Phi_\beta(P)$ (respectively, $\Phi^\pm_\beta(P)$), $\Phi_\beta(Q)$ (respectively, $\Phi^\pm_\beta(Q)$) *are also connectable, and* $\Phi_\beta(P * Q) = \Phi_\beta(P) * \Phi_\beta(Q)$ (respectively, $\Phi^\pm_\beta(P * Q) = \Phi^\pm_\beta(P) * \Phi^\pm_\beta(Q)$).

Proof. Again, only the proof of the case $P, Q \in \mathcal{A}_n$ will be shown, as the proof of the other two cases is similar.

We first prove $\Phi_\beta(a_{ij}^{x+y}) = \Phi_\beta(a_{ik}^x) * \Phi_\beta(a_{kj}^y)$.

By Theorem 2.8,

$$\Phi_\beta(a_{ik}^x) = \Phi_\beta(\psi(\gamma_{ik}^x)) = \psi\tilde{\Phi}_\beta(\gamma_{ik}^x), \quad \Phi_\beta(a_{kj}^y) = \psi\tilde{\Phi}_\beta(\gamma_{kj}^y). \tag{2.7}$$

By Proposition 2.11,

$$\tilde{\Phi}_\beta(\gamma_{ik}^x) * \tilde{\Phi}_\beta(\gamma_{kj}^y) = \tilde{\Phi}_\beta(\gamma_{ik}^x * \gamma_{kj}^y) = \tilde{\Phi}_\beta(\gamma_{ij}^{x+y}). \tag{2.8}$$

By Proposition 2.14 and Eq. (2.7),

$$\psi(\tilde{\Phi}_\beta(\gamma_{ik}^x) * \tilde{\Phi}_\beta(\gamma_{kj}^y)) = \psi(\tilde{\Phi}_\beta(\gamma_{ik}^x)) * \psi(\tilde{\Phi}_\beta(\gamma_{kj}^y)) = \Phi_\beta(a_{ik}^x) * \Phi_\beta(a_{kj}^y). \tag{2.9}$$

Also by Theorem 2.8,

$$\psi(\tilde{\Phi}_\beta(\gamma_{ij}^{x+y})) = \Phi_\beta(\psi(\gamma_{ij}^{x+y})) = \Phi_\beta(a_{ij}^{x+y}). \tag{2.10}$$

Combining Eqs. (2.8)–(2.10), we get $\Phi_\beta(a_{ij}^{x+y}) = \Phi_\beta(a_{ik}^x) * \Phi_\beta(a_{kj}^y)$.

In general, let P, Q be as described in Definition 2.13, then for $\beta \in \mathcal{C}_n$, $\Phi_\beta(P) = \sum_{x\in\mathbb{Z}}\sum_{i=1}^n \Phi_\beta(P_i^x)\Phi_\beta(a_{ik}^x)$, and $\Phi_\beta(Q) = \sum_{y\in\mathbb{Z}}\sum_{j=1}^n \Phi_\beta(a_{kj}^y)\Phi_\beta(Q_j^y)$. Since $\Phi_\beta(a_{ik}^x)$ and $\Phi_\beta(a_{kj}^y)$ are connectable, $\Phi_\beta(P)$ and $\Phi_\beta(Q)$ are also connectable. Moreover,

$$\begin{aligned}
\Phi_\beta(P) * \Phi_\beta(Q) &= \sum_{x,y\in\mathbb{Z}}\sum_{i,j=1}^n \Phi_\beta(P_i^x)\{\Phi_\beta(a_{ik}^x) * \Phi_\beta(a_{kj}^y)\}\Phi_\beta(Q_j^y) \\
&= \sum_{x,y\in\mathbb{Z}}\sum_{i,j=1}^n \Phi_\beta(P_i^x)\Phi_\beta(a_{ij}^{x+y})\Phi_\beta(Q_j^y) \\
&= \Phi_\beta\left(\sum_{x,y\in\mathbb{Z}}\sum_{i,j=1}^n P_i^x a_{ij}^{x+y} Q_j^y\right) \\
&= \Phi_\beta(P * Q).
\end{aligned}$$

□

Remark 2.16. We will identify $\tilde{\mathcal{A}}_n$ with $\mathcal{A}_n$, $\tilde{\mathcal{A}}_n^\pm$ with $\mathcal{A}_n^\pm$, γ_{ij}^x with a_{ij}^x via the corresponding isomorphisms and identify $\tilde{\Phi}_\beta$ with Φ_β, $\tilde{\Phi}_\beta^\pm$ with $\Phi_\beta^\pm$, respectively. A useful picture to keep in mind is as follows. a_{ij}^x is the left arc diagram described in Fig. 6. The action Φ_β (respectively, $\Phi_\beta^\pm$) of β on some curve is to slide that curve along the parallel copy of the braid diagram that represents β (respectively, $\epsilon^\pm(\beta)$) up to $D_n \times \{1\}$ (respectively, $D_n^\pm \times \{1\}$).

3. The Framed Knot Invariant

From now on, we will assume the closure of $\beta \in C_n$ is a knot in $S^1 \times S^2$.

In this section, we first give the definition of the framed knot invariant. Since the knot invariant looks complicated at first glance, we will compute some examples after the definition. We then proceed to give some ancillary results, and finally prove the invariance under Markov moves.

3.1. *Definition of the invariant*

Here are some notations we will use to define the invariant.

Let $M_\infty(\mathcal{A}_n)$ denote the set of $\infty \times \infty$ matrices with elements in $\mathcal{A}_n$, namely, the rows and columns of a matrix in $M_\infty(\mathcal{A}_n)$ are both indexed by integers. We call a matrix *row-finite* if there are only finitely many nonzero entries in each row. A *column-finite* matrix is defined analogously. If M, N are two matrices in $\mathcal{M}_\infty(\mathcal{A}_n)$, in general the multiplication of them is not well-defined. However, if M is *row-finite* or N is *column-finite*, then MN is well-defined. And the associativity is satisfied whenever multiplications make sense. Throughout the paper, the matrices always satisfy the above condition when they are multiplied together, and for $x, y \in \mathbb{Z}$, we will use M^{xy} to refer to the (x, y)-entry of M. We will also use an element $c \in A_n$ to represent the scalar matrix in $M_\infty(\mathcal{A}_n)$ which has entry c on the diagonal and 0 elsewhere. Let $M_n(M_\infty(\mathcal{A}_n))$ denote the set of $n \times n$ matrices with entries in $M_\infty(\mathcal{A}_n)$.

Recall $\epsilon^\pm : \mathcal{C}_n \to \mathcal{C}_{n+1}$ are the two embeddings, and for $\beta \in \mathcal{C}_n$, $(\Phi^-_\beta)_{|\mathcal{A}_n} = \Phi_\beta = (\Phi^+_\beta)_{|\mathcal{A}_n}$.

It can be derived from either the algebraic description (Sec. 2.2.1) or the topological interpretation (Sec. 2.2.2) of the actions that for $1 \le i \le n, x \in \mathbb{Z}$, $\Phi^-_\beta(a^x_{i,n+1})$ can be written as a finite linear combination of $a^z_{k,n+1}, 1 \le k \le n, z \in \mathbb{Z}$ with coefficients in $\mathcal{A}_n$. A similar argument holds for $\Phi^-_\beta(a^x_{n+1,i}), \Phi^+_\beta(a^x_{i,0}), \Phi^+_\beta(a^x_{0,i})$. For example, $\Phi^+_\beta(a^x_{0,i})$ is a finite linear combination of $a^z_{0,k}$ with coefficients in $\mathcal{A}_n$ multiplied on the right. Explicitly, this is how we define $\Phi^{-L}_\beta, \Phi^{-R}_\beta, \Phi^{+L}_\beta, \Phi^{+R}_\beta \in M_n(M_\infty(\mathcal{A}_n))$ below.

For each $\beta \in \mathcal{C}_n, 1 \le i, j \le n, x, y \in \mathbb{Z}$, define

$$\Phi^-_\beta(a^x_{i,n+1}) = \sum_{k=1}^n \sum_{z\in\mathbb{Z}} (\Phi^{-L}_\beta)^{xz}_{ik} a^z_{k,n+1},$$

$$\Phi^-_\beta(a^y_{n+1,j}) = \sum_{k=1}^n \sum_{z\in\mathbb{Z}} a^z_{n+1,k} (\Phi^{-R}_\beta)^{zy}_{kj},$$

$$\Phi^+_\beta(a^x_{i,0}) = \sum_{k=1}^n \sum_{z\in\mathbb{Z}} (\Phi^{+L}_\beta)^{xz}_{ik} a^z_{k,0},$$

$$\Phi^+_\beta(a^y_{0,j}) = \sum_{k=1}^n \sum_{z\in\mathbb{Z}} a^z_{0,k} (\Phi^{+R}_\beta)^{zy}_{kj},$$

where $(\Phi^{-L}_\beta)^{xz}_{ik}$ is the (x, z)-entry of the $\infty \times \infty$ matrix $(\Phi^{-L}_\beta)_{ik}$ which is the (i, k)-entry of the $n \times n$ matrix Φ^{-L}_β. So we have $\Phi^{-L}_\beta \in M_n(M_\infty(\mathcal{A}_n))$. Similarly, we have $\Phi^{-R}_\beta, \Phi^{+L}_\beta, \Phi^{+R}_\beta \in M_n(M_\infty(\mathcal{A}_n))$.

Define $a_{ij} \in M_\infty(\mathcal{A}_n)$ by $(a_{ij})^{xy} = a^{x+y}_{ij}$ and define $A \in M_n(M_\infty(\mathcal{A}_n))$ by $A_{ij} = a_{ij}$.

Lemma 3.1. *For $\beta \in \mathcal{C}_n, 1 \le i, j \le n$, $(\Phi_\beta^{-L})_{ij}, (\Phi_\beta^{+L})_{ij}$ are row-finite and $(\Phi_\beta^{-R})_{ij}, (\Phi_\beta^{+R})_{ij}$ are column-finite.*

Proof. These are direct consequences of the definitions. □

Lemma 3.1 is used to validate the matrix multiplications involving $\Phi_\beta^{-L}, \Phi_\beta^{+L}, \Phi_\beta^{-R}$ and Φ_β^{+R} in the rest of the paper. For instance, the product $\Phi_\beta^{-L} A \Phi_\beta^{-R}$ is well-defined.

Remark 3.2. Actually, $(\Phi_\beta^{+L})_{ij}, (\Phi_\beta^{+R})_{ij}$ are both *row-finite* and *column-finite*. This is due to a careful inspection of the action Φ_β^+. This property will not be used though.

For $1 \le p, q \le n, f \in \mathbb{Z}$, let $\Lambda_{f;p,q} \in M_n(M_\infty(\mathcal{A}_n))$ be the diagonal matrix with the (i,i)-th entry $\lambda^{\delta_{i,p}} \mu^{-f\delta_{i,q}}$, $1 \le i \le n$.

Definition 3.3. Let $\beta \in \mathcal{C}_n, 1 \le p, q \le n, f \in \mathbb{Z}$, then $HC_0(\beta; f; p, q)$ is defined to be the R-algebra $\mathcal{A}_n$ modulo the two sided ideal $\mathcal{I}_{\beta;f;p,q}$ generated by the entries of the entries of following matrices:

$$A - \Lambda_{f;p,q}\Phi_\beta^{-L} A, \quad A - A\Phi_\beta^{-R}\Lambda_{f;p,q}^{-1},$$
$$A - \Lambda_{f;p,q}\Phi_\beta^{+L} A, \quad A - A\Phi_\beta^{+R}\Lambda_{f;p,q}^{-1}.$$

Remark 3.4. (1) For a matrix $M \in M_n(M_\infty(\mathcal{A}_n))$, the phrase "the entries of the entries of M" is really awkward. We will use "the elements of M" to stand for "the entries of the entries of M".

(2) Note that

$$\begin{aligned}
(\Lambda_{p,q;f}\Phi_\beta^{-L} A)_{ij}^{xy} &= \sum_{k=1}^{n}\sum_{z\in\mathbb{Z}} \lambda^{\delta_{i,p}}\mu^{-f\delta_{i,q}} (\Phi_\beta^{-L})_{ik}^{xz} A_{kj}^{zy} \\
&= \sum_{k=1}^{n}\sum_{z\in\mathbb{Z}} \lambda^{\delta_{i,p}}\mu^{-f\delta_{i,q}} ((\Phi_\beta^{-L})_{ik}^{xz} a_{k,n+1}^{z}) * a_{n+1,j}^{y} \\
&= \lambda^{\delta_{i,p}}\mu^{-f\delta_{i,q}} \Phi_\beta^{-}(a_{i,n+1}^{x}) * a_{n+1,j}^{y}.
\end{aligned}$$

Since $A_{ij}^{xy} = a_{ij}^{x+y} = a_{i,n+1}^x * a_{n+1,j}^y$, the relations in $\mathcal{I}_{\beta;f;p,q}$ are the same as the following:

$$\begin{aligned}
&a_{i,n+1}^x * a_{n+1,j}^y - \lambda^{\delta_{i,p}}\mu^{-f\delta_{i,q}} \Phi_\beta^{-}(a_{i,n+1}^x) * a_{n+1,j}^y, \\
&a_{i,n+1}^x * a_{n+1,j}^y - \lambda^{-\delta_{j,p}}\mu^{f\delta_{j,q}} a_{i,n+1}^x * \Phi_\beta^{-}(a_{n+1,j}^y), \\
&a_{i,0}^x * a_{0,j}^y - \lambda^{\delta_{i,p}}\mu^{-f\delta_{i,q}} \Phi_\beta^{+}(a_{i,0}^x) * a_{0,j}^y, \\
&a_{i,0}^x * a_{0,j}^y - \lambda^{-\delta_{j,p}}\mu^{f\delta_{j,q}} a_{i,0}^x * \Phi_\beta^{+}(a_{0,j}^y), \quad \forall 1 \le i, j \le n, x, y \in \mathbb{Z}.
\end{aligned}$$

For $\beta \in C_n$, it has a natural action by permutation on the set $\{1, \dots, n\}$. Our convention here is that the braid diagram always goes upward, and if the ith strand ends at the jth position, then $\beta(i) = j$.

Lemma 3.5. *For $\beta \in \mathcal{C}_n, 1 \leq p, q \leq n, f \in \mathbb{Z}$, we have $HC_0(\beta; f; p, q) \simeq HC_0(\beta; f; \beta(p), q) \simeq HC_0(\beta; f; p, \beta(q))$.*

Proof. Define $\psi : HC_0(\beta; f; p, q) \to HC_0(\beta; f; \beta(p), q)$ by $\psi(a^x_{ij}) = \lambda^{-\delta_{i,\beta(p)}} \times a^x_{ij} \lambda^{\delta_{j,\beta(p)}}$. We need to check that ψ sends $\mathcal{I}_{\beta;f;p,q}$ to $\mathcal{I}_{\beta;f;\beta(p),q}$.

Note that $\Phi^-_\beta(a^x_{i,n+1}) * a^y_{n+1,j}$ can be written as a non-commutative polynomial in which each monomial is of the form $a^{x_1}_{\beta(i),i_1} a^{x_2}_{i_1,i_2} \cdots a^{x_{k+1}}_{i_k,j}$, thus we have

$$\begin{aligned} &\psi(a^{x+y}_{ij} - \lambda^{\delta_{i,p}} \mu^{-f\delta_{i,q}} \Phi^-_\beta(a^x_{i,n+1}) * a^y_{n+1,j}) \\ &\quad = \lambda^{-\delta_{i,\beta(p)}} a^{x+y}_{ij} \lambda^{\delta_{j,\beta(p)}} - \lambda^{\delta_{i,p}} \mu^{-f\delta_{i,q}} \lambda^{-\delta_{\beta(i),\beta(p)}} \Phi^-_\beta(a^x_{i,n+1}) * a^y_{n+1,j} \lambda^{\delta_{j,\beta(p)}} \\ &\quad = \lambda^{-\delta_{i,\beta(p)}} (a^{x+y}_{ij} - \lambda^{\delta_{i,\beta(p)}} \mu^{-f\delta_{i,q}} \Phi^-_\beta(a^x_{i,n+1}) * a^y_{n+1,j}) \lambda^{\delta_{j,\beta(p)}} \in \mathcal{I}_{\beta;f;\beta(p),q}. \end{aligned}$$

The other three relations in $\mathcal{I}_{\beta;f;p,q}$ can be shown analogously that they are mapped under ψ to $\mathcal{I}_{\beta;f;\beta(p),q}$. Thus the map ψ is well-defined. It follows directly from the definition that ψ is a bijection.

The isomorphism $HC_0(\beta; f; p, q) \simeq HC_0(\beta; f; p, \beta(q))$ can be defined in a similar way by mapping a^x_{ij} to $\mu^{f\delta_{i,\beta(p)}} a^x_{ij} \mu^{-f\delta_{j,\beta(p)}}$. □

Corollary 3.6. *If the closure of $\beta \in \mathcal{C}_n$ is a knot in $S^1 \times S^2$, then $HC_0(\beta; f; p, q)$ is independent of the values of p, q.*

From Definition 3.3, $HC_0(\beta; f; p, p)$ can be obtained from $HC_0(\beta; 0; p, p)$ by replacing λ by $\lambda\mu^{-f}$. We will use the notations $HC_0(\beta; f; p) = HC_0(\beta; f; p, p)$, $HC_0(\beta; f) = HC_0(\beta; f; 1, 1)$ and $HC_0(\beta) = HC_0(\beta; 0; 1, 1)$. By Corollary 3.6, $HC_0(\beta; f; p)$ is independent of the choice of p, so we have $HC_0(\beta; f) \simeq HC_0(\beta; f; p)$ for any p.

The following theorem is our main result.

Theorem 3.7. *Let $\beta, \alpha \in C_n, f \in \mathbb{Z}$ such that the closure of β in $S^1 \times S^2$ is a knot, then we have the following algebra isomorphisms*:

(1) $HC_0(\beta; f) \simeq HC_0(\alpha^{-1}\beta\alpha; f)$;
(2) $HC_0(\beta; f) \simeq HC_0(\epsilon^-(\beta)\alpha_n; f - 1) \simeq HC_0(\epsilon^-(\beta)\alpha_n^{-1}; f + 1)$;
(3) $HC_0(\beta; f) \simeq HC_0(\epsilon^+(\beta)\alpha_1; f - 1) \simeq HC_0(\epsilon^+(\beta)\alpha_1^{-1}; f + 1)$.

The proof of the theorem will be given in Sec. 3.4.

Endow $S^1 \times S^2$ with the standard orientation. Let K be a framed oriented knot in $S^1 \times S^2$ with l, m the homotopy classes of the longitude and the meridian of K in $\pi_1(S^1 \times S^2 \backslash K)$. The orientations of K and $S^1 \times S^2$ determine the meridian class m uniquely. More precisely, let $\nu(K)$ be a tubular neighborhood of K, which is homeomorphic to $K \times D^2$. Choose an orientation on D^2 so that the homeomorphism of $\nu(K)$ with $K \times D^2$ is orientation preserving. Then for any $z \in K$, the image of $z \times \partial D^2$ under the homeomorphism determines the meridian class. Assume K is represented by the closure of a braid $\beta \in \mathcal{C}_n$, and that $l = [\hat{\beta}']m^f$, where β' is a parallel push-off copy of β and $[\hat{\beta}']$ is the homotopy class represented by the closure of β', then $HC_0(K;l)$ is defined to be $HC_0(\beta;f)$.

Corollary 3.8. *$HC_0(K;l)$ as an R-algebra is a framed knot invariant for knots in $S^1 \times S^2$.*

Proof. For a braid diagram $\beta \in \mathcal{C}_n$, let β' be the parallel push-off copy of β. Then we have $[\hat{\beta}']m^{\pm 1} = [(\epsilon^+\widehat{(\beta)\alpha_n^{\pm 1}})']$, $[\hat{\beta}']m^{\pm 1} = [(\epsilon^-\widehat{(\beta)\alpha_n^{\pm 1}})']$ and for any $\alpha \in \mathcal{C}_n$, we have $[\hat{\beta}'] = [(\widehat{\alpha^{-1}\beta\alpha})']$. $\square$

Remark 3.9. The invariant $HC_0(K;l)$ is conjectured to be the zeroth knot contact homology of K, which is defined to be the zeroth Legendrian contact homology of Λ_K in $ST^*(S^1 \times S^2)$, where $ST^*(S^1 \times S^2)$ is the unit cotangent bundle of $S^1 \times S^2$ and Λ_K is the unit conormal bundle of K. As this paper is not relevant to proving this conjecture, readers should just treat HC_0 purely as a name.

3.2. *Examples*

Before proving invariance, we first look at some examples.

Example 3.10. (1) **Unknot.** The most simple example is the unknot represented by the identity element e in $\mathcal{C}_1$. We compute $HC_0(e;f)$ for $f \in \mathbb{Z}$. In this case, it is straightforward that $\Phi_e^{+L}, \Phi_e^{+R}, \Phi_e^{-L}, \Phi_e^{-R}$ are all identity matrices, thus all the relations in $\mathcal{I}_{e;f;1,1}$ become $(1-\lambda\mu^{-f})a_{11}^x$, and so $HC_0(e;f) \simeq R\langle a_{11}^x, x \in \mathbb{Z}\rangle / \langle (1-\lambda\mu^{-f})a_{11}^x \rangle$.

(2) $\widehat{\boldsymbol{\alpha_0^2}}$**.** Set $\beta = \alpha_0^2, \Lambda = \Lambda_{\beta;0;1,1}$. We first compute $\Phi_\beta^{+L}, \Phi_\beta^{+R}$. Direct calculations show that $\Phi_\beta^+(a_{10}^x) = \mu^2 a_{10}^{x-2}, \Phi_\beta^+(a_{01}^y) = \mu^{-2}a_{01}^{y+2}$. Thus we have $(\Phi_\beta^{+L})_{11}^{xy} = \mu^2\delta_{x-2,y}, (\Phi_\beta^{+R})_{11}^{xy} = \mu^{-2}\delta_{x-2,y}$, and therefore $(\Lambda\Phi_\beta^{+L}A)_{11}^{xy} = \lambda\mu^2 a_{11}^{x+y-2}, (A\Phi_\beta^{+R} \times \Lambda^{-1})_{11}^{xy} = (\lambda\mu^2)^{-1}a_{11}^{x+y+2}$. So the third and fourth relation defining $\mathcal{I}_{\beta;0;1,1}$ both are $a_{11}^{x+2} - \lambda\mu^2 a_{11}^x, x \in \mathbb{Z}$.

Now we compute $\Phi_\beta^{-L}, \Phi_\beta^{-R}$. By definition, $\Phi_{\alpha_0}^-(a_{11}^x) = a_{11}^x, \Phi_{\alpha_0}^-(a_{12}^x) = -\mu a_{12}^{x-1} + \frac{1}{\Gamma}a_{11}^x a_{12}^{-1}$. Therefore,

$$\Phi_{\alpha_0^2}^-(a_{12}^x) = -\mu\Phi_{\alpha_0}^-(a_{12}^{x-1}) + \frac{1}{\Gamma}\Phi_{\alpha_0}^-(a_{11}^x)\Phi_{\alpha_0}^-(a_{12}^{-1})$$

$$= \mu^2 a_{12}^{x-2} - \frac{\mu}{\Gamma}a_{11}^{x-1}a_{12}^{-1} - \frac{\mu}{\Gamma}a_{11}^x a_{12}^{-2} + \frac{1}{\Gamma^2}a_{11}^x a_{11}^{-1} a_{12}^{-1}.$$

By Part (2) of Remark 3.4,

$(\Lambda\Phi_\beta^{-L}A)_{11}^{xy} - A_{11}^{xy} = \lambda(\mu^2 a_{11}^{x+y-2} - \frac{\mu}{\Gamma}a_{11}^{x-1}a_{11}^{y-1} - \frac{\mu}{\Gamma}a_{11}^x a_{11}^{y-2} + \frac{1}{\Gamma^2}a_{11}^x a_{11}^{-1}a_{11}^{y-1}) - a_{11}^{x+y}$.

Similarly,
$(A\Phi_\beta^{-R}\Lambda^{-1})_{11}^{xy} - A_{11}^{xy} = (\lambda\mu^2)^{-1}(a_{11}^{x+y+2} - \frac{1}{\Gamma}a_{11}^{x+1}a_{11}^{y+1} - \frac{1}{\Gamma}a_{11}^{x+2}a_{11}^{y} + \frac{1}{\Gamma^2}a_{11}^{x+1} \times a_{11}^1 a_{11}^y) - a_{11}^{x+y}$.

Since we have $a_{11}^{x+2} - \lambda\mu^2 a_{11}^x$, then the above two relations can be simplified as $a_{11}^{x-1}a_{11}^{y-1} + a_{11}^x a_{11}^{y-2} - \frac{1}{\Gamma\mu}a_{11}^x a_{11}^{-1} a_{11}^{y-1}$ and

$$a_{11}^{x-1}a_{11}^{y-1} + a_{11}^x a_{11}^{y-2} - \frac{1}{\Gamma}a_{11}^{x-1}a_{11}^1 a_{11}^{y-2}.$$

And only parities of x and y will make a difference in the above two relations.

Direct calculation shows that $HC_0(\beta) \simeq R[X]/\langle(1-\mu)X, X^2 - \Gamma^2\lambda(1+\mu)^2\rangle$. Replacing λ by $\lambda\mu^{-f}$, we obtain $HC_0(\beta; f)$.

It will be shown in Sec. 4.2 that $\widehat{\alpha_0^2}$ is a particular knot in a large family of knots, namely the torus knots. Explicitly, it is the $(1,2)$-knot. See Sec. 4.2 for a definition of torus knots and more examples.

3.3. *Properties of* $\Phi^{\pm L}, \Phi^{\pm R}$

We give several propositions which will be used in proving the invariance of $HC_0(K; l)$. A similar version of these propositions are proved in [7] where the author defined the HC_0 invariant for knots in S^3.

If ϕ is an algebra morphism from $\mathcal{A}_n$ to $\mathcal{A}_n$, and $M \in M_n(M_\infty(\mathcal{A}_n))$, we denote by $\phi(M)$ or $M(\phi)$ the matrix obtained from M by replacing each a_{ij}^x by $\phi(a_{ij}^x)$. Recall in last subsection, we defined the four matrices $\Phi_\beta^{-L}, \Phi_\beta^{-R}, \Phi_\beta^{+L}, \Phi_\beta^{+R} \in M_n(M_\infty(\mathcal{A}_n))$ for $\beta \in \mathcal{C}_n$.

Proposition 3.11. *Let $\beta_1, \beta_2 \in \mathcal{C}_n$ be two braids, then we have*

$$\begin{aligned}
\Phi_{\beta_1\beta_2}^{-L} &= \Phi_{\beta_2}^{-L}(\Phi_{\beta_1})\Phi_{\beta_1}^{-L},\\
\Phi_{\beta_1\beta_2}^{-R} &= \Phi_{\beta_1}^{-R}\Phi_{\beta_2}^{-R}(\Phi_{\beta_1}),\\
\Phi_{\beta_1\beta_2}^{+L} &= \Phi_{\beta_2}^{+L}(\Phi_{\beta_1})\Phi_{\beta_1}^{+L},\\
\Phi_{\beta_1\beta_2}^{+R} &= \Phi_{\beta_1}^{+R}\Phi_{\beta_2}^{+R}(\Phi_{\beta_1}).
\end{aligned}$$

Proof. The proof of the four equalities are straightforward and completely analogous, so we will just prove the first one.

By definition, $\Phi_{\beta_2}^{-}(a_{i,n+1}^x) = \sum_{k=1}^n \sum_{z\in\mathbb{Z}} (\Phi_{\beta_2}^{-L})_{ik}^{xz} a_{k,n+1}^z$. Thus,

$$\begin{aligned}
\Phi_{\beta_1\beta_2}^{-}(a_{i,n+1}^x) &= \Phi_{\beta_1}^{-}\Phi_{\beta_2}^{-}(a_{i,n+1}^x)\\
&= \sum_{k=1}^n \sum_{z\in\mathbb{Z}} \Phi_{\beta_1}^{-}((\Phi_{\beta_2}^{-L})_{ik}^{xz})\Phi_{\beta_1}^{-}(a_{k,n+1}^z)
\end{aligned}$$

$$= \sum_{k,j=1}^{n} \sum_{z,y\in\mathbb{Z}} \Phi_{\beta_2}^{-L}(\Phi_{\beta_1})_{ik}^{xz}(\Phi_{\beta_1}^{-L})_{kj}^{zy} a_{j,n+1}^{y}$$

$$= \sum_{j=1}^{n} \sum_{y\in\mathbb{Z}} (\Phi_{\beta_2}^{-L}(\Phi_{\beta_1})\Phi_{\beta_1}^{-L})_{ij}^{xy} a_{j,n+1}^{y}$$

On the other hand, by definition, $\Phi_{\beta_1\beta_2}^{-}(a_{i,n+1}^{x}) = \sum_{j=1}^{n}\sum_{z\in\mathbb{Z}}(\Phi_{\beta_1\beta_2}^{-L})_{ij}^{xy} a_{j,n+1}^{y}$.

Therefore, we have $(\Phi_{\beta_2}^{-L}(\Phi_{\beta_1})\Phi_{\beta_1}^{-L})_{ij}^{xy} = (\Phi_{\beta_1\beta_2}^{-L})_{ij}^{xy}$. □

Let $I_n \in M_n(M_\infty(\mathcal{A}_n))$ be the identity matrix, i.e. $(I_n)_{ij}^{xy} = \delta_{i,j}\delta_{x,y}$. Then apparently, for a trivial braid $\beta \in \mathcal{C}_n$, $\Phi_\beta^{-L}, \Phi_\beta^{-R}, \Phi_\beta^{+L}, \Phi_\beta^{+R}$ are all equal to I_n. Therefore, we have the following corollary.

Corollary 3.12. *For any braid $\beta \in \mathcal{C}_n$, $\Phi_\beta^{-L}, \Phi_\beta^{-R}, \Phi_\beta^{+L}, \Phi_\beta^{+R}$ are all invertible. Explicitly,*

$$(\Phi_\beta^{-L})^{-1} = \Phi_{\beta^{-1}}^{-L}(\Phi_\beta), \quad (\Phi_\beta^{-R})^{-1} = \Phi_{\beta^{-1}}^{-R}(\Phi_\beta),$$
$$(\Phi_\beta^{+L})^{-1} = \Phi_{\beta^{-1}}^{+L}(\Phi_\beta), \quad (\Phi_\beta^{+R})^{-1} = \Phi_{\beta^{-1}}^{+R}(\Phi_\beta).$$

Proof. In Proposition 3.11, set $\beta_1 = \beta, \beta_2 = \beta^{-1}$. □

Proposition 3.13. *For any $\beta \in \mathcal{C}_n$, we have $\Phi_\beta(A) = \Phi_\beta^{-L} A \Phi_\beta^{-R} = \Phi_\beta^{+L} A \Phi_\beta^{+R}$.*

Proof. By Proposition 3.11, it suffices to show the above equations hold for any $\alpha_k \in \mathcal{C}_n$, which can be verified directly.

Here we provide another way to prove it.

$$\Phi_\beta(A_{ij}^{xy}) = \Phi_\beta^{-}(a_{ij}^{x+y}) = \Phi_\beta^{-}(a_{i,n+1}^{x} * a_{n+1,j}^{y})$$

$$\overset{Proposition\ 2.15}{=} \Phi_\beta^{-}(a_{i,n+1}^{x}) * \Phi_\beta^{-}(a_{n+1,j}^{y})$$

$$= \left(\sum_{k=1}^{n}\sum_{z\in\mathbb{Z}}(\Phi_\beta^{-L})_{ik}^{xz} a_{k,n+1}^{z}\right) * \left(\sum_{k'=1}^{n}\sum_{z'\in\mathbb{Z}} a_{n+1,k'}^{z'}(\Phi_\beta^{-R})_{k'j}^{z'y}\right)$$

$$= \sum_{k,k'=1}^{n}\sum_{z,z'\in\mathbb{Z}}(\Phi_\beta^{-L})_{ik}^{xz} a_{kk'}^{z+z'}(\Phi_\beta^{-R})_{k'j}^{z'y}$$

$$= \sum_{k,k'=1}^{n}\sum_{z,z'\in\mathbb{Z}}(\Phi_\beta^{-L})_{ik}^{xz} A_{kk'}^{zz'}(\Phi_\beta^{-R})_{k'j}^{z'y}$$

$$= (\Phi_\beta^{-L} A \Phi_\beta^{-R})_{ij}^{xy}$$

The other equation can be proved analogously. □

Corollary 3.14. *For $\beta \in \mathcal{C}_n, 1 \leq p,q \leq n, f \in \mathbb{Z}$, the elements of $A - \Lambda_{f;p,q}\Phi_\beta(A)\Lambda_{f;p,q}^{-1}$ are in $\mathcal{I}_{\beta;f;p,q}$. More generally, if $b = a_{i_1,i_2}^{x_1}a_{i_2,i_3}^{x_2}\cdots a_{i_k,i_{k+1}}^{x_k}, c_i = \lambda^{\delta_{i,p}}\mu^{-f\delta_{i,q}}$, then $b - c_{i_1}\Phi_\beta(b)c_{i_{k+1}}^{-1}$ is in $\mathcal{I}_{\beta;f;p,q}$.*

Proof. Set $\Lambda = \Lambda_{f;p,q}$. Then

$$A - \Lambda\Phi_\beta(A)\Lambda^{-1} = A - \Lambda\Phi_\beta^{-L}A\Phi_\beta^{-R}\Lambda^{-1} = A - \Lambda\Phi_\beta^{-L}A + \Lambda\Phi_\beta^{-L}(A - A\Phi_\beta^{-R}\Lambda^{-1}).$$

The elements of the right-hand side of the above equation are in $\mathcal{I}_{\beta;f;p,q}$, which implies the first part of the corollary. The more general statement in the corollary is then a direct consequence. □

3.4. *Invariance proof*

In this subsection, we prove Theorem 3.7. The three parts in the theorem correspond to the three types of Markov moves introduced in Theorem 2.3. In the following three subsections, we prove each part of the theorem, respectively.

3.4.1. *Invariance under Markov move* I

Let $\tilde{\beta} = \alpha^{-1}\beta\alpha, \alpha, \beta \in \mathcal{C}_n, f \in \mathbb{Z}$, and define $m = \alpha^{-1}(1)$. Set $\Lambda_i = \Lambda_{f;i,i}$. We define an isomorphism $\varphi : HC_0(\tilde{\beta}; f; m) \to HC_0(\beta; f; 1)$ by specifying the image of the generators.

$$\varphi(A) := \Phi_\alpha(A), \text{ i.e.} \varphi(a_{ij}^x) := \Phi_\alpha(a_{ij}^x).$$

We need to show $\varphi(\mathcal{I}_{\tilde{\beta};f;m,m}) \subset \mathcal{I}_{\beta;f;1,1}$.

First of all, by using Proposition 3.11, we have

$$\Phi_\alpha(\Phi_{\alpha^{-1}\beta\alpha}^{-L}) = \Phi_\alpha(\Phi_{\beta\alpha}^{-L}(\Phi_{\alpha^{-1}})\Phi_{\alpha^{-1}}^{-L}) = \Phi_{\beta\alpha}^{-L}\Phi_{\alpha^{-1}}^{-L}(\Phi_\alpha) = \Phi_\alpha^{-L}(\Phi_\beta)\Phi_\beta^{-L}\Phi_{\alpha^{-1}}^{-L}(\Phi_\alpha),$$

Therefore, we have

$$\begin{aligned}
\varphi(A - \Lambda_m\Phi_{\tilde{\beta}}^{-L}A) &= \varphi(A) - \Lambda_m\varphi(\Phi_{\tilde{\beta}}^{-L})\varphi(A) \\
&= \Phi_\alpha(A) - \Lambda_m\Phi_\alpha(\Phi_{\alpha^{-1}\beta\alpha}^{-L})\Phi_\alpha(A) \\
&= \Phi_\alpha(A) - \Lambda_m\Phi_\alpha^{-L}(\Phi_\beta)\Phi_\beta^{-L}\Phi_{\alpha^{-1}}^{-L}(\Phi_\alpha)\Phi_\alpha^{-L}A\Phi_\alpha^{-R} \\
&= \Phi_\alpha^{-L}A\Phi_\alpha^{-R} - \Lambda_m\Phi_\alpha^{-L}(\Phi_\beta)\Phi_\beta^{-L}A\Phi_\alpha^{-R} \\
&= (\Phi_\alpha^{-L} - \Lambda_m\Phi_\alpha^{-L}(\Phi_\beta)\Lambda_1^{-1})A\Phi_\alpha^{-R} \\
&\quad + \Lambda_m\Phi_\alpha^{-L}(\Phi_\beta)\Lambda_1^{-1}(A - \Lambda_1\Phi_\beta^{-L}A)\Phi_\alpha^{-R}.
\end{aligned}$$

Since $(\Phi_\alpha^{-L})_{ij}^{xy}$ is a non-commutative polynomial in which each monomial is of the form $a_{\alpha(i),j_1}^{x_1}a_{j_1,j_2}^{x_2}\cdots a_{j_{k-1},j}^{x_k}$, and note that $\delta_{i,m} = \delta_{\alpha(i),1}$, then $(\Phi_\alpha^{-L} - \Lambda_m\Phi_\alpha^{-L}(\Phi_\beta)\Lambda_1^{-1})_{ij}^{xy}$ is a sum of polynomials of the form $a_{\alpha(i),j_1}^{x_1}a_{j_1,j_2}^{x_2}\cdots a_{j_{k-1},j}^{x_k} - (\lambda\mu^{-f})^{\delta_{\alpha(i),1}}\Phi_\beta(a_{\alpha(i),j_1}^{x_1}a_{j_1,j_2}^{x_2}\cdots a_{j_{k-1},j}^{x_k})(\lambda\mu^{-f})^{-\delta_{j,1}}$, which, by Corollary 3.14, is in $\mathcal{I}_{\beta;f;1,1}$.

Since elements of $A - \Lambda_1 \Phi_{\beta}^{-L} A$ are also in $\mathcal{I}_{\beta;f;1,1}$, this implies $\varphi(A - \Lambda_m \Phi_{\tilde{\beta}}^{-L} A) \subset \mathcal{I}_{\beta;f;1,1}$.

The proofs of the other three relations $\varphi(A - A\Phi_{\tilde{\beta}}^{-R}\Lambda_m^{-1})$, $\varphi(A - \Lambda_m \Phi_{\tilde{\beta}}^{+L} A)$, $\varphi(A - A\Phi_{\tilde{\beta}}^{+R}\Lambda_m^{-1})$ can be done similarly.

This shows $\varphi(\mathcal{I}_{\tilde{\beta}}; f; m, m) \subset \varphi(\mathcal{I}_{\beta}; f; 1, 1)$ and thus φ induces a well-defined map $HC_0(\tilde{\beta}; f) \to HC_0(\beta; f)$. In a similar way, we can define the inverse map $HC_0(\beta; f) \to HC_0(\tilde{\beta}; f)$ by sending A to $\Phi_{\alpha^{-1}}(A)$ and show that it is well-defined. Thus φ is an isomorphism.

3.4.2. *Invariance under Markov move* II

For any $\beta \in \mathcal{C}_n, f \in \mathbb{Z}$ let $\tilde{\beta} = \epsilon^-(\beta)\alpha_n$. We show $HC_0(\tilde{\beta}; f) \simeq HC_0(\beta; f+1)$.

Remark 3.15. The proof of $HC_0(\epsilon^-(\beta)\alpha_n^{-1}; f+1) \simeq HC_0(\beta; f)$ is completely analogous. To save space, we omit its proof here.

Define $\varphi : HC_0(\tilde{\beta}; f; n+1) \to HC_0(\beta; f+1; n)$,

$$\varphi(a_{ij}^x) = \begin{cases} a_{nn}^x & i = n+1, j = n+1, \\ \mu a_{nj}^x & i = n+1, j \leq n, \\ \mu^{-1} a_{in}^x & i \leq n, j = n+1, \\ a_{ij}^x & i \leq n, j \leq n. \end{cases} \tag{3.1}$$

The verification that φ maps $\mathcal{I}_{\tilde{\beta};f;n+1,n+1}$ to $\mathcal{I}_{\beta;f+1;n,n}$ consists of direct but long calculations. we will only show $\varphi(a_{i,j}^{x+y} - (\lambda\mu^{-f})^{\delta_{i,n+1}} \Phi_{\tilde{\beta}}^{-}(a_{i,n+2}^x) * a_{n+2,j}^y) \in \mathcal{I}_{\beta;f+1;n,n}$. The other relations can be proven similarly.

Set $c = \lambda\mu^{-f}, \mathcal{I} = \mathcal{I}_{\beta;f+1;n,n}$.

Case 1: $i = n+1, j \leq n$.

$$\begin{aligned} \varphi(a_{n+1,j}^{x+y} - c\Phi_{\tilde{\beta}}^{-}(a_{n+1,n+2}^x) * a_{n+2,j}^y) &= \mu a_{n,j}^{x+y} - c\varphi(\Phi_{\epsilon^-(\beta)}^{-}(a_{n,n+2}^x) * a_{n+2,j}^y) \\ &= \mu a_{n,j}^{x+y} - c\Phi_{\beta}^{-}(a_{n,n+1}^x) * a_{n+1,j}^y \\ &= \mu(a_{n,j}^{x+y} - \lambda\mu^{-f-1}\Phi_{\beta}^{-}(a_{n,n+1}^x) * a_{n+1,j}^y) \in \mathcal{I}. \end{aligned}$$

Note that here we used the fact that $\Phi_{\epsilon^-(\beta)}^{-}(a_{n,n+2}^x) * a_{n+2,j}^y = \Phi_{\beta}^{-}(a_{n,n+1}^x) * a_{n+1,j}^y \in \mathcal{A}_{n+1}$.

Case 2: $i = n, j \leq n$.

$$\begin{aligned} &a_{n,j}^{x+y} - \Phi_{\tilde{\beta}}^{-}(a_{n,n+2}^x) * a_{n+2,j}^y \\ &\quad = a_{n,j}^{x+y} - \Phi_{\epsilon^-(\beta)}^{-}(-a_{n+1,n+2}^x + \frac{1}{\Gamma\mu} a_{n+1,n}^0 a_{n,n+2}^x) * a_{n+2,j}^y \end{aligned}$$

$$= a^{x+y}_{n,j} - (-a^x_{n+1,n+2} + \frac{1}{\Gamma\mu}\Phi^-_{\epsilon^-(\beta)}(a^0_{n+1,n})\Phi^-_{\epsilon^-(\beta)}(a^x_{n,n+2})) * a^y_{n+2,j}$$

$$= a^{x+y}_{n,j} + a^{x+y}_{n+1,j} - \frac{1}{\Gamma\mu}\Phi_{\epsilon^-(\beta)}(a^0_{n+1,n})\Phi^-_{\epsilon^-(\beta)}(a^x_{n,n+2}) * a^y_{n+2,j}$$

$$= a^{x+y}_{n,j} + a^{x+y}_{n+1,j} - \frac{1}{\Gamma\mu}\Phi^-_{\beta}(a^0_{n+1,n})\Phi^-_{\beta}(a^x_{n,n+1}) * a^y_{n+1,j}.$$

Since $\varphi(\Phi^-_\beta(a^0_{n+1,n})) = \mu a^0_{n,n+1} * \Phi^-_\beta(a^0_{n+1,n}) = ca^0_{nn} \,(\mathrm{mod}\,\mathcal{I})$,

$$\varphi(a^{x+y}_{n,j} - \Phi^-_{\tilde\beta}(a^x_{n,n+2}) * a^y_{n+2,j})$$

$$= (1+\mu)a^{x+y}_{nj} - \frac{1}{\Gamma\mu}c(1+\mu)\Gamma\Phi^-_\beta(a^x_{n,n+1}) * a^y_{n+1,j}(\mathrm{mod}\,\mathcal{I})$$

$$= (1+\mu)(a^{x+y}_{nj} - \lambda\mu^{-f-1}\Phi^-_\beta(a^x_{n,n+1}) * a^y_{n+1,j})(\mathrm{mod}\,\mathcal{I}) = 0(\mathrm{mod}\,\mathcal{I}).$$

Case 3: $i \le n-1, j \le n$.

$$\begin{aligned}\varphi(a^{x+y}_{i,j} - \Phi^-_{\tilde\beta}(a^x_{i,n+2}) * a^y_{n+2,j}) &= \varphi(a^{x+y}_{i,j} - \Phi^-_{\epsilon^-(\beta)}(a^x_{i,n+2}) * a^y_{n+2,j})\\ &= \varphi(a^{x+y}_{i,j} - \Phi^-_\beta(a^x_{i,n+1}) * a^y_{n+1,j})\\ &= a^{x+y}_{i,j} - \Phi^-_\beta(a^x_{i,n+1}) * a^y_{n+1,j} \in \mathcal{I}.\end{aligned}$$

Case 4: $j = n+1$. The proof is the same as the above three cases except each expression is multiplied by an overall scalar μ^{-1}.

This finishes the verification. One can also define a map $\theta : HC_0(\beta; f+1; n) \to HC_0(\tilde\beta; f; n+1)$ sending a^x_{ij} to a^x_{ij}, and show that it is well-defined. Clearly we have $\varphi\theta = \mathrm{Id}$. To show $\theta\varphi = \mathrm{Id}$, we need to prove in $HC_0(\tilde\beta; f; n+1)$, we have the equalities $a^x_{i,n} = \mu a^x_{i,n+1}, a^x_{n,j} = \mu^{-1}a^x_{n+1,j}$ for $1 \le i,j \le n+1$.

In $HC_0(\tilde\beta; f; n+1)$, we have, for $1 \le i,j \le n+1$,

$$\begin{aligned}a^x_{i,n+1} &= a^x_{i,n+2} * \Phi^-_{\tilde\beta}(a^0_{n+2,n+1})c^{-1} = a^x_{i,n+2} * \Phi^-_{\epsilon^-(\beta)}(a^0_{n+2,n})c^{-1}\\ &= a^x_{i,n+1} * \Phi^-_\beta(a^0_{n+1,n})c^{-1}, \quad a^x_{n+1,j} = \Phi^-_{\tilde\beta}(a^0_{n+1,n+2}) * a^x_{n+2,j}c\\ &= \Phi^-_{\epsilon^-(\beta)}(a^0_{n,n+2}) * a^x_{n+2,j}c = \Phi^-_\beta(a^0_{n,n+1}) * a^x_{n+1,j}c.\end{aligned}$$

In the above two equalities, set $i = j = n+1, x = 0$, then we get $\Phi^-_\beta(a^0_{n+1,n}) = a^0_{n+1,n+1}\lambda c = (1+\mu)\Gamma c$ and $\Phi^-_\beta(a^0_{n,n+1}) = (1+\mu)\Gamma c^{-1}$.

Then,

$$\begin{aligned}a^x_{i,n} &= a^x_{i,n+2} * \Phi^-_{\tilde\beta}(a^0_{n+2,n})\\ &= a^x_{i,n+2} * \Phi^-_{\epsilon^-(\beta)}(-a^0_{n+2,n+1} + \frac{1}{\Gamma}a^0_{n+2,n}a^0_{n,n+1})\end{aligned}$$

$$= a^x_{i,n+2} * \left(-a^0_{n+2,n+1} + \frac{1}{\Gamma}\Phi^-_{\epsilon^-(\beta)}(a^0_{n+2,n})\Phi^-_{\epsilon^-(\beta)}(a^0_{n,n+1})\right)$$

$$= -a^x_{i,n+1} + \frac{1}{\Gamma}a^x_{i,n+1} * \Phi^-_\beta(a^0_{n+1,n})\Phi^-_\beta(a^0_{n,n+1})$$

$$= -a^x_{i,n+1} + \frac{1}{\Gamma}ca^x_{i,n+1}(1+\mu)\Gamma c^{-1} = \mu a^x_{i,n+1},$$

and similarly,

$$\begin{aligned} a^x_{n,j} &= \Phi^-_{\tilde{\beta}}(a^0_{n,n+2}) * a^x_{n+2,j} \\ &= \Phi^-_{\epsilon^-(\beta)}\left(-a^0_{n+1,n+2} + \frac{1}{\Gamma\mu}a^0_{n+1,n}a^0_{n,n+2}\right) * a^x_{n+2,j} \\ &= \left(-a^0_{n+1,n+2} + \frac{1}{\Gamma\mu}\Phi^-_{\epsilon^-(\beta)}(a^0_{n+1,n})\Phi^-_{\epsilon^-(\beta)}(a^0_{n,n+2})\right) * a^x_{n+2,j} \\ &= -a^x_{n+1,j} + \frac{1}{\Gamma\mu}\Phi^-_\beta(a^0_{n+1,n})\Phi^-_\beta(a^0_{n,n+1}) * a^x_{n+1,j} \\ &= -a^x_{n+1,j} + \frac{1}{\Gamma\mu}(1+\mu)\Gamma cc^{-1}a^x_{n+1,j} \\ &= \mu^{-1}a^x_{n+1,j}. \end{aligned}$$

Therefore, we showed $\theta\varphi = \mathrm{Id}$. Together with the fact that $\varphi\theta = \mathrm{Id}$, we know φ is an isomorphism.

3.4.3. *Invariance under Markov move* III

Recall that D_n is the unit disk with $n+1$ punctures $p, p_1, \ldots, p_n$ centered at the origin of the complex plane. To be more precise, let p be the origin and the coordinate of p_i be $\frac{i}{n+1}$. We define a map $r : D_n \to D_n$ by $r(z) = \frac{\bar{z}}{|z|} - \bar{z}$. Namely, r is a reflection about the x-axis followed by another reflection about the circle centered at the origin with radius $\frac{1}{2}$. Note that $r^2 = \mathrm{Id}$. Also $r \times \mathrm{Id}$ defines a map on $X = D_n \times [0,1]$, which will still be denoted by r.

Since $\mathcal{C}_n$ is the braid group on the punctured disk D_n inside X, the map r on X induces a group isomorphism from $\mathcal{C}_n$ to itself. Explicitly, the isomorphism, also denoted by r, is given by:

$$r(\alpha_i) = \begin{cases} (\alpha_{n-1}\cdots\alpha_1\alpha_0\alpha_1\cdots\alpha_{n-1})^{-1} & i = 0, \\ \alpha_{n-i} & 1 \le i \le n-1. \end{cases} \tag{3.2}$$

Lemma 3.16. *The map r defined above from $\mathcal{C}_n$ to $\mathcal{C}_n$ is a group isomorphism and $r^2 = \mathrm{Id}$.*

Proof. This can be verified purely algebraically. □

Also recall that $q_1, \ldots, q_n$ are n points with the coordinate $\frac{i}{n+1} - \epsilon$ for some tiny $\epsilon > 0$. And $Q_n = \{q_i, 1 \le i \le n\}$, $\mathcal{Q}_n = \{\gamma : [0,1] \to D_n | \gamma \text{ is continuous}, \gamma(0), \gamma(1) \in Q_n\} \sim$. Let $q'_{n+1-i} = r(q_i)$, which has the coordinate $\frac{n+1-i}{n+1} + \epsilon$, and let $Q'_n = \{q'_i, 1 \le i \le n\}$. It should be clear that in the definition of $\tilde{\mathcal{A}}_n$, if we replace q_i by q'_i, insist that the curves start and end at q'_i, and change the "skein" relations accordingly, then we get the same algebra.

For a curve $\gamma \in \mathcal{Q}_n$ from q_i to q_j, $r(\gamma)$ is a curve from q'_{n+1-i} to q'_{n+1-j}. The map r also preserves the "skein" relations in Fig. 5 that defines $\tilde{A}_n$. Thus r induces an algebra isomorphism from $\tilde{A}_n$ to $\tilde{A}_n$.

Explicitly, the map $r : \tilde{A}_n \to \tilde{A}_n$ is given by Fig. 12.

Remark 3.17. r also extends to a bijection from $\mathcal{Q}_n^+$ to $\mathcal{Q}_n^-$ by furthermore requiring that p_0 is mapped to p_{n+1}. And r maps the "skein" relations that define $\mathcal{A}_n^+$ to the corresponding "skein" relations that define $\mathcal{A}_n^-$. Consequently, we get an isomorphism $r : \mathcal{A}_n^+ \to \mathcal{A}_n^-$. Note that the inverse map is also induced by r that maps $\mathcal{Q}_n^-$ to $\mathcal{Q}_n^+$. For this reason, we will denote the inverse map also by r. In summary, r is an isomorphism between $\mathcal{A}_n^+$ and $\mathcal{A}_n^-$, which restricts to an isomorphism on $\mathcal{A}_n$ and which has square Id.

Lemma 3.18. *If $P, Q \in \mathcal{A}_n^\pm$ are connectable, then $r(P)$, $r(Q)$ are connectable, and $r(P * Q) = r(P) * r(Q)$.*

Proof. This follows from the geometrical interpretation of a_{ij}^x and the map r. □

Lemma 3.19. *If β is a braid in $\mathcal{C}_n$, then we have $r \circ \Phi_\beta = \Phi_{r(\beta)} \circ r$. More generally, we have $r \circ \Phi_\beta^- = \Phi_{r(\beta)}^+ \circ r$.*

Proof. It is possible, though tedious, to prove it algebraically. For example, it suffices to prove the case for $\beta = \alpha_k^{\pm 1}$ acting on a_{ij}^x. Here we give another geometric proof which makes the statement in the lemma almost trivial. Recall that the isomorphism $r : \mathcal{C}_n \to \mathcal{C}_n$ is induced by the homeomorphism $r \times \mathrm{Id} : D_n \times I \to D_n \times I$. By Remark 2.16, $\Phi_\beta(\gamma_{ij}^x)$ can be obtained as the curve by sliding γ_{ij}^x in $D_n \times \{0\}$

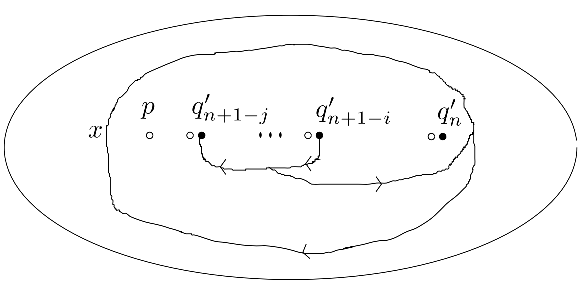
$r(a_{ij}^x) =$ 

Fig. 12. $r(a_{ij}^x)$.

along the parallel copy diagram β' up to $D_n \times \{1\}$. The map $r \times \mathrm{Id}$ maps γ^x_{ij} to $r(\gamma^x_{ij})$, $\Phi_\beta(\gamma^x_{ij})$ to $r \circ \Phi_\beta(\gamma^x_{ij})$, and β to $r(\beta)$. Thus $r \circ \Phi_\beta(\gamma^x_{ij})$ is obtained by sliding $r(\gamma^x_{ij})$ along the parallel copy braid diagram $r(\beta)'$, and therefore $\Phi_{r(\beta)} \circ r(\gamma^x_{ij}) = r \circ \Phi_\beta(\gamma^x_{ij})$.

The more general equation can be proved analogously by using Remark 2.16 and Lemma 3.21. $\square$

Proposition 3.20. *For $\beta \in \mathcal{C}_n, f \in \mathbb{Z}$ the map $r : \mathcal{A}_n \to \mathcal{A}_n$ induces an isomorphism from $HC_0(\beta; f; 1)$ to $HC_0(r(\beta); f; n)$.*

Proof. It suffices to show r maps $I_{\beta;f;1,1}$ to $I_{r(\beta);f;n,n}$. Set $c = \lambda\mu^{-f}$.

$$\begin{aligned} r((\Lambda_{f;1,1}\Phi^{-L}_\beta A)^{xy}_{ij}) &= r(c^{\delta_{i,1}}\Phi^-_\beta(a^x_{i,n+1}) * a^y_{n+1,j}) \\ &= c^{\delta_{i,1}}(r \circ \Phi^-_\beta(a^x_{i,n+1})) * r(a^y_{n+1,j}) \\ &= c^{\delta_{n+1-i,n}}(\Phi^+_{r(\beta)} \circ r(a^x_{i,n+1})) * r(a^y_{n+1,j}). \end{aligned}$$

The first identity in the above equation is by the argument in Part (2) of Remark 3.4, the second identity is by Lemma 3.18, and the third by Lemma 3.19.

Assume $r(a^x_{i,n+1}) = \sum P^z_k a^z_{k0}, r(a^y_{n+1,j}) = \sum a^{z'}_{0k'} Q^{z'}_{k'}$, where $P^z_k, Q^{z'}_{k'}$ are elements in $\mathcal{A}_n$. Then

$$\begin{aligned} &r((A - \Lambda_{f;1,1}\Phi^{-L}_\beta A)^{xy}_{ij}) \\ &\quad = \sum P^z_k a^{zz'}_{kk'} Q^{z'}_{k'} - c^{\delta_{n+1-i,n}}\Phi_{r(\beta)}(P^z_k)\Phi^+_{r(\beta)}(a^z_{k0}) * a^{z'}_{0k'} Q^{z'}_{k'} \\ &\quad = \sum (P^z_k - c^{\delta_{n+1-i,n}}\Phi_{r(\beta)}(P^z_k)c^{-\delta_{k,n}}) a^{zz'}_{kk'} Q^{z'}_{k'} \\ &\qquad + c^{\delta_{n+1-i,n}}\Phi_{r(\beta)}(P^z_k)c^{-\delta_{k,n}}(a^{zz'}_{kk'} - c^{\delta_{k,n}}\Phi^+_{r(\beta)}(a^z_{k0}) * a^{z'}_{0k'}) Q^{z'}_{k'}. \end{aligned}$$

Note that P^z_k is a sum of monomials of the form $a^{x_1}_{n+1-i,i_1} a^{x_2}_{i_1,i_2} \cdots a^{x_m}_{i_{m-1},k}$, then $P^z_k - c^{\delta_{n+1-i,n}}\Phi_{r(\beta)}(P^z_k)c^{-\delta_{k,n}}$ is in $I_{r(\beta);f;n,n}$ by Corollary 3.14.

Then it follows that $r((A - \Lambda_{f;1,1}\Phi^{-L}_\beta A)^{xy}_{ij})$ is in $I_{r(\beta);f;n,n}$.

The other relations are proved basically in the same way. And thus we showed r is well-defined. The fact that r is an isomorphism is direct to check. $\square$

Now we prove $HC_0(\beta; f)$ is invariant under Markov move III. A key observation is the following commuting diagram.

$$\begin{tikzcd} \mathcal{C}_n \arrow[r, "\epsilon^+"] \arrow[d, "r"] & \mathcal{C}_{n+1} \arrow[d, "r"] \\ \mathcal{C}_n \arrow[r, "\epsilon^-"] & \mathcal{C}_{n+1} \end{tikzcd} \tag{3.3}$$

Lemma 3.21. *The above diagram commutes, namely $r \circ \epsilon^+ = \epsilon^- \circ r : \mathcal{C}_n \to \mathcal{C}_{n+1}$.*

Proof. We only need to check the above equation on generators.

$$\begin{aligned} r\epsilon^+(\alpha_0) &= r(\alpha_1\alpha_0\alpha_1) = \alpha_n(\alpha_n\cdots\alpha_1\alpha_0\alpha_1\cdots\alpha_n)^{-1}\alpha_n \\ &= (\alpha_{n-1}\cdots\alpha_1\alpha_0\alpha_1\cdots\alpha_{n-1})^{-1} = \epsilon^- r(\alpha_0). \end{aligned}$$

For $i \geq 1$, $r\epsilon^+(\alpha_i) = r(\alpha_{i+1}) = \alpha_{n-i} = \epsilon^- r(\alpha_i)$. □

Let $\beta \in \mathcal{C}_n, f \in \mathbb{Z}$, then $r(\epsilon^+(\beta)\alpha_1^{\pm 1}) = r(\epsilon^+(\beta))r(\alpha_1^{\pm 1}) = \epsilon^-(r(\beta))\alpha_n^{\pm 1}$. Therefore,

$$\begin{aligned} HC_0(\epsilon^+(\beta)\alpha_1^{\pm 1}; f) &\simeq HC_0(r(\epsilon^+(\beta)\alpha_1^{\pm 1}); f) = HC_0(\epsilon^-(r(\beta))\alpha_n^{\pm 1}; f) \\ &\simeq HC_0(r(\beta); f \pm 1) \simeq HC_0(\beta; f \pm 1). \end{aligned}$$

The first and last isomorphisms above are due to Proposition 3.20 and the second isomorphism is the invariance isomorphism under Markov move II.

Now we finished showing $HC_0(\beta; f)$ is invariant under Markov move III.

4. Properties of the Invariant

4.1. *Symmetries of the invariant*

In Proposition 3.20, it was proved that for a braid $\beta \in \mathcal{C}_n$, we have $HC_0(\beta; f) \simeq HC_0(r(\beta); f)$. Here we show the relation between $HC_0(\beta; f)$ and $HC_0(\beta^{-1}; f)$.

Proposition 4.1. *Let $\beta \in \mathcal{C}_n, f \in \mathbb{Z}$, then $HC_0(\beta^{-1}; f)$ is isomorphic to $HC_0(\beta; -f)$ with λ replaced by λ^{-1}.*

Proof. Let $HC_0'(\beta; -f)$ be the algebra obtained from $HC_0(\beta; -f)$ by replacing λ by λ^{-1}. We define the isomorphism $HC_0(\beta^{-1}; f) \to HC_0'(\beta; -f)$ to be the one induced by Φ_β. We need to check Φ_β maps $\mathcal{I}_{\beta^{-1}; f; 1,1}$ to $\mathcal{I}_{\beta; -f; 1,1}$ with λ replaced by λ^{-1}. Set $\Lambda = \Lambda_{f;1,1}$, and note that Λ^{-1} is exactly the matrix $\Lambda_{-f;1,1}$ with λ replaced by λ^{-1}.

$$\begin{aligned} \Phi_\beta(\Lambda\Phi^{+L}_{\beta^{-1}}A - A) &= \Lambda\Phi^{+L}_{\beta^{-1}}(\Phi_\beta)\Phi_\beta(A) - \Phi_\beta(A) \\ &= \Lambda\Phi^{+L}_{\beta^{-1}}(\Phi_\beta)\Phi^{+L}_\beta A\Phi^{+R}_\beta - \Phi^{+L}_\beta A\Phi^{+R}_\beta \\ &= \Lambda A\Phi^{+R}_\beta - \Phi^{+L}_\beta A\Phi^{+R}_\beta \\ &= \Lambda(A - \Lambda^{-1}\Phi^{+L}_\beta A)\Phi^{+R}_\beta \end{aligned}$$

The second equality is by Proposition 3.13 and the third one is by Corollary 3.12.

The other three relations can be proved analogously that they are mapped to 0 in $HC_0'(\beta; -f)$. Therefore, Φ_β induces a well-defined algebra map from $HC_0(\beta^{-1}; f)$ to $HC_0'(\beta; -f)$.

In a similar way, one can check that $\Phi_{\beta^{-1}}$ induces an algebra map from $HC_0'(\beta; -f)$ to $HC_0(\beta^{-1}; f)$. Therefore, we have $HC_0(\beta^{-1}; f) \simeq HC_0'(\beta; -f)$. □

4.2. *Torus Knots*

In this subsection, we study some properties of the torus knots in $S^1 \times S^2$.

Let C be the equator of S^2, then $S^1 \times C$ is a torus which bounds two solid tori in $S^1 \times S^2$, with $z_0 \times C$ being the meridian and $S^1 \times z_1$ the longitude, for some $z_0 \in S^1, z_1 \in C$. In [1], a knot in $S^1 \times S^2$ is called a torus knot if it can be isotoped to a knot in $S^1 \times C$. Fix a meridian M and a longitude L in $S^1 \times C$, and let p, q be two relatively prime integers. A (p,q)-knot in $S^1 \times S^2$ is a knot which can be isotoped to $pM + qL$ in $S^1 \times C$. In general, for a knot K and a framing l, $HC_0(K;l)$ may not be finitely generated as an R-algebra. However, we show below that for torus knots, the invariant indeed is always finitely generated.

Theorem 4.2. *Let K be a (p,q)-knot in $S^1 \times S^2$ with framing l where p, q are relatively prime integers, then $HC_0(K;l)$ is finitely generated as an R-algebra. Moreover, the minimum number of algebra generators is no more than $q-1$.*

Proof. By Remark 2.4, a (p,q)-knot is represented by the braid $\beta(p,q) = (\alpha_0 \cdots \alpha_{p-1})^q$. See Fig. 13 for a picture of $(3,2)$-knot. For simplicity, we still use β to denote $\beta(p,q)$. Also for reasons that will become clear below, we use the notation $b^x_{ij} = a^x_{i+1,j+1}$. Assume $HC_0(K;l) = HC_0(\beta;f) = \mathcal{A}_p/\mathcal{I}_{\beta;f;1,1}$, and set $c = \lambda\mu^{-f}$. One can check that the following equation holds:

$$\Phi^+_{\beta(p,1)}(a^x_{i0}) = \begin{cases} a^x_{i+1,0} & 1 \le i \le p-1, \\ \mu a^{x-1}_{1,0} & i = p. \end{cases} \tag{4.1}$$

Then we have $\Phi^+_{\beta(p,q)}(a^x_{i0}) = \mu^{\lfloor \frac{i-1+q}{p} \rfloor} a^{x-\lfloor \frac{i-1+q}{p} \rfloor}_{(i-1+q)\,(\mathrm{mod}\,p)+1,0}$. Using b^x_{ij} to replace $a^x_{i+1,j+1}$, we get a simpler expression $\Phi^+_{\beta(p,q)}(b^x_{i,-1}) = \mu^{\lfloor \frac{i+q}{p} \rfloor} b^{x-\lfloor \frac{i+q}{p} \rfloor}_{(i+q)\,(\mathrm{mod}\,p),-1}$.

Thus by Remark 3.4(2), the third relation that defines $\mathcal{I}_{\beta;f;1,1}$ is

$$b^x_{ij} - \mu^{\lfloor \frac{i+q}{p} \rfloor} c^{\delta_{i,0}} b^{x-\lfloor \frac{i+q}{p} \rfloor}_{(i+q)\,(\mathrm{mod}\,p),j}, 0 \le i, j \le p-1, x \in \mathbb{Z}. \tag{4.2}$$

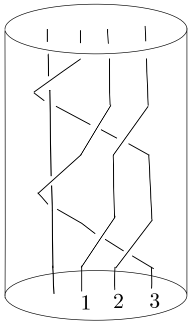


Fig. 13. $(3,2)$-knot.

Similarly, the fourth relation that defines $\mathcal{I}_{\beta;f;1,1}$ is

$$b_{ij}^x - \mu^{-\lfloor\frac{j+q}{p}\rfloor}c^{-\delta_{j,0}}b_{i,(j+q)\,(\mathrm{mod}\,p)}^{x+\lfloor\frac{j+q}{p}\rfloor}, 0 \le i,j \le p-1, x \in \mathbb{Z}. \tag{4.3}$$

Define $g(i,k) := \sum_{r=0}^{k-1}\lfloor\frac{(i+rq)\,(\mathrm{mod}\,p)+q}{p}\rfloor, h(i,k) := \sum_{r=0}^{k-1}\delta_{(i+rq)\,(\mathrm{mod}\ p),0}, 0 \le i \le p-1, k \ge 1$, and define $g(i,0) := 0, h(i,0) := 0$.

It is elementary to verify that $g(i,k) = \lfloor\frac{k}{p}\rfloor q + g(i, k \bmod p)$ and $h(i,k) = \lfloor\frac{k}{p}\rfloor + h(i, k \bmod p)$, and in $HC_0(\beta; f; 1, 1)$, we have the equalities $b_{ij}^x = \mu^{g(i,k)}c^{h(i,k)}b_{(i+kq)\,(\mathrm{mod}\,p),j}^{x-g(i,k)} = \mu^{-g(j,k)}c^{-h(j,k)}b_{i,(j+kq)\,(\mathrm{mod}\,p)}^{x+g(j,k)}, \forall k \ge 0$. Especially, we have $b_{ij}^x = \mu^{g(i,p)}c^{h(i,p)}b_{ij}^{x-g(i,p)} = \mu^q c b_{ij}^{x-q}$, so b_{ij}^x is periodic, up to a scalar, in x with period equal to q.

Let k_1, k_2 be any numbers that satisfy $k_1 q\,(\mathrm{mod}\,p) = i, k_2 q\,(\mathrm{mod}\,p) = j$, then $b_{ij}^x = \mu^{g(0,k_2)-g(0,k_1)}c^{h(0,k_2)-h(0,k_1)}b_{00}^{x+g(0,k_1)-g(0,k_2)}$, and $b_{00}^{x+q} = \mu^q c b_{00}^x$. Thus all the b_{ij}^x's are completely determined by $b_{00}^0 = (1+\mu)\Gamma, b_{00}^1, \ldots, b_{00}^{q-1}$ and the condition that $b_{00}^{x+q} = \mu^q c b_{00}^x$. So $HC_0(\beta; f)$ is finitely generated and $\{b_{00}^x, 1 \le x \le q-1\}$ is a set of generators. □

At the end of this subsection, let us compute some examples of torus knots.

Example 4.3. (1) ***(p,1)*-knot**. The $(p,1)$-knot is represented by the braid $\alpha_0 \cdots \alpha_{p-1}$. By Markov II in Theorem 2.3, this braid has the same closure as that of α_0. Set $\beta = \alpha_0 \in \mathcal{C}_1, \Lambda = \Lambda_{0;1,1}, f = 0$. By the proof of Theorem 4.2, $a_{11}^{x+1} = \lambda\mu a_{11}^x$. Since $a_{11}^0 = (1+\mu)\Gamma$, we have $a_{11}^x = (1+\mu)\Gamma(\lambda\mu)^x$.

By definition, $\Phi_\beta^-(a_{12}^x) = -\mu a_{12}^{x-1} + \frac{1}{\Gamma}a_{11}^x a_{12}^{-1}$, thus $(\Lambda\Phi_\beta^{-L}A - A)_{11}^{xy} = -\lambda\mu a_{11}^{x+y-1} + \frac{\lambda}{\Gamma}a_{11}^x a_{12}^{y-1} - a_{11}^{x+y}$. The second relation can be calculated analogously. By using the fact that $a_{11}^x = (1+\mu)\Gamma(\lambda\mu)^x$, we see that $HC_0(\alpha_0) \simeq R/\langle\mu^2 - 1\rangle$.

(2) ***(p,2)*-knot**. By [1, Proposition 2.2], all the $(p,2)$-knots are equivalent to each other with p odd. This can also be seen directly by Markov moves. Thus, we only need to compute the $(1,2)$-knot, which is represented by $\beta = \alpha_0^2$. It was shown in the second example in Sec. 3.2 that $HC_0(\alpha_0^2) \simeq R[X]/\langle(1-\mu)X, X^2 - \Gamma^2\lambda(1+\mu)^2\rangle$.

(3) ***(p,3)*-knot**. Again by [1, Proposition 2.2], there are two classes of knots of this type. A representative of each class could be chosen as $(1,3)$-knot and $(2,3)$-knot. Here we only compute $HC_0(\alpha_0^3)$. Since the calculations are not difficult but tedious, we just present the result obtained by computer packages. $HC_0(\alpha_0^3) \simeq R\langle X, Y\rangle/\langle Y^2 - \Gamma\lambda\mu^4(1+\mu^2)X, X^2 - \Gamma(1+\mu^{-2})Y, (1+\mu^2)(XY - YX), -\mu^2 XY + YX + \Gamma^2\lambda\mu^4(\mu^2 - 1)\rangle$.

4.3. *Local knots*

Throughout this subsection, Γ is set to be -1. A knot is called *local* if it is contained in a 3-ball. Thus a knot in $S^1 \times S^2$ is local if and only if it can be represented as the closure of a braid which does not contain α_0 or α_0^{-1}, i.e. a braid in $\mathcal{B}_n = \langle\alpha_1, \ldots, \alpha_{n-1}\rangle \subset \mathcal{C}_n$ for some $n \ge 1$.

Let $\beta \in \mathcal{B}_n \subset \mathcal{C}_n$, and let $hc_0(\beta)$ denote the zeroth framed knot contact homology of β defined in [9]. Then we have the following decompositions for the HC_0 invariant of local knots, which relates our invariant to hc_0.

Theorem 4.4. *Let $\beta \in \mathcal{B}_n$ be a braid such that its closure is a local knot in $S^1 \times S^2$, then there exists a strictly ascending sequence of subalgebras of $HC_0(\beta)$, $H_0 \subsetneqq H_1 \subsetneqq H_2 \subsetneqq \ldots$, such that $HC_0(\beta) = \bigcup_{m\geq 0} H_m$, and that $H_0 \simeq hc_0(\beta)$ if we make the change of variable $\mu \to \mu^{-1}$ in $hc_0(\beta)$. In particular, $HC_0(\beta)$ is an infinitely generated algebra.*

Proof. Set $\Lambda = \Lambda_{0;1,1}$.

Since $\beta \in \mathcal{B}_n$ does not contain α_0 or α_0^{-1}, the actions Φ_β^- and Φ_β^+ are determined by Eq. 2.2. It can be derived from Eq. 2.2 that $\Phi_\beta^-(a_{i,n+1}^x) = \Phi_\beta^-(a_{i,n+1}^0) * a_{n+1,n+1}^x$ and that $\Phi_\beta^+(a_{i,0}^x) = \Phi_\beta^-(a_{i,n+1}^x) * a_{n+1,0}^0 = \Phi_\beta^-(a_{i,n+1}^0) * a_{n+1,0}^x$. Therefore, $(\Phi_\beta^{+L}A)_{ij}^{xy} = \Phi_\beta^+(a_{i,0}^x) * a_{0,j}^y = \Phi_\beta^-(a_{i,n+1}^x) * a_{n+1,j}^y = (\Phi_\beta^{-L}A)_{ij}^{xy}$, i.e. $\Phi_\beta^{+L}A = \Phi_\beta^{-L}A$. Similarly, we have $A\Phi_\beta^{+R} = A\Phi_\beta^{-R}$. So to compute $HC_0(\beta)$, we only need to consider the relations $A - \Lambda\Phi_\beta^{-L}A$ and $A - A\Phi_\beta^{-R}\Lambda^{-1}$.

$(\Lambda\Phi_\beta^{-L}A)_{ij}^{xy} = \lambda^{\delta_{i,1}}\Phi_\beta^-(a_{i,n+1}^x) * a_{n+1,j}^y = \lambda^{\delta_{i,1}}\Phi_\beta^-(a_{i,n+1}^0) * a_{n+1,j}^{x+y} = \lambda^{\delta_{i,1}}\Phi_\beta^-(a_{i,n+1}^0) * a_{n+1,j}^0 * a_{j,j}^{x+y} = (\Lambda\Phi_\beta^{-L}A)_{ij}^{00} * a_{j,j}^{x+y}$.

Thus, $A_{ij}^{xy} - (\Lambda\Phi_\beta^{-L}A)_{ij}^{xy} = (A_{ij}^{00} - (\Lambda\Phi_\beta^{-L}A)_{ij}^{00}) * a_{jj}^{x+y}$. Similarly, $A_{ij}^{xy} - (A\Phi_\beta^{-R}\Lambda^{-1})_{ij}^{xy} = a_{ii}^{x+y} * (A_{ij}^{00} - (A\Phi_\beta^{-R}\Lambda^{-1})_{ij}^{00})$.

For each non-negative integer m, let

$$E_m = \mathbb{Z}\langle a_{ij}^x, 1 \leq i, j \leq n, |x| \leq m\rangle,$$

and let $H_m = E_m/J_m$, where J_m is the idea of E_m generated by the following elements:

$$(A_{ij}^{00} - (\Lambda\Phi_\beta^{-L}A)_{ij}^{00}) * a_{jj}^x, \quad a_{ii}^x * (A_{ij}^{00} - (A\Phi_\beta^{-R}\Lambda^{-1})_{ij}^{00}), \quad 1 \leq i, j \leq n, |x| \leq m.$$

For each m, one can define the algebra morphism $\iota_m : H_m \to HC_0(\beta)$, such that $\iota_m(a_{ij}^x) = a_{ij}^x$, $|x| \leq m$, and the algebra morphism $\pi_m : HC_0(\beta) \to H_m$, such that $\pi_m(a_{ij}^x) = a_{ij}^x$ if $|x| \leq m$, and that $\pi_m(a_{ij}^x) = 0$ otherwise. The maps ι_m and π_m are clearly both well-defined, and $\pi_m\iota_m = Id$. Thus, ι_m is injective. Identifying H_m with its image $\iota_m(H_m)$ in $HC_0(\beta)$, we get $HC_0(\beta) = \bigcup_{m\geq 0} H_m$.

Next, we show H_m is a proper subalgebra of H_{m+1}.

The map π_m, restricting on H_{m+1}, sends a_{ij}^{m+1} to 0, and for $|x| \leq m$ sends each a_{ij}^x to a_{ij}^x. Thus $a_{ij}^{m+1} \in H_m$ if and only if $a_{ij}^{m+1} = 0$. However, by Proposition 4.6 which is to be proved in Sec. 4.4, if one sets $\lambda = \mu = 1$, then there is a $\mathbb{Z}$-algebra morphism from $HC_0(\beta)$ to $\mathbb{Z}$ mapping each a_{ij}^x to -2. [c] This implies none of the a_{ij}^x's is 0 in $HC_0(\beta)$ when λ, μ are set to 1, which furthermore implies the a_{ij}^x's are not 0 in the original $HC_0(\beta)$. Therefore, a_{ij}^{m+1} is not contained in H_m, and H_m is a proper subalgebra of H_{m+1}.

[c]Note that in this subsection we have set $\Gamma = -1$.

Lastly we show that $H_0 \simeq hc'_0(\beta)$, where $hc'_0(\beta)$ is $hc_0(\beta)$ with μ replaced by μ^{-1}.

By Eq. (2.2), the action of Φ_β on E_0 is exactly the same as the braid action given in [9] if we set $\Gamma = -1$, and make the change of variables as follows: $a^0_{ij} = \mu a_{ij}$ if $i > j$ and $a^0_{ij} = a_{ij}$ otherwise. The readers should be warned that here a_{ij} is the symbol used in [9], but not the $\infty \times \infty$ matrix we defined before. In the language of [9], a^0_{ij} is the same as $\mu a'_{ij}$ in that paper.[d] Moreover,

$$\begin{aligned}\Phi^-_\beta(a^0_{i,n+1}) * a^0_{n+1,j} &= \Phi^-_\beta(a_{i,n+1}) * a^0_{n+1,j} = \sum((\Phi^L_\beta)_{ik} a_{k,n+1}) * a^0_{n+1,j} \\ &= \sum((\Phi^L_\beta)_{ik} a^0_{k,n+1}) * a^0_{n+1,j} = \sum(\Phi^L_\beta)_{ik} a^0_{k,j}.\end{aligned}$$

Then we have $A^{00}_{ij} - (\Lambda\Phi^{-L}_\beta A)^{00}_{ij} = a^0_{ij} - \lambda^{\delta_{i,1}}\Phi^-_\beta(a^0_{i,n+1}) * a^0_{n+1,j} = a^0_{ij} - \lambda^{\delta_{i,1}}\sum(\Phi^L_\beta)_{ik}a^0_{k,j}$, which is exactly μ times the (i,j)-entry of $A - \Lambda\Phi^L_\beta A$ defined in [9]. Similarly, $A^{00}_{ij} - (A\Phi^{-R}_\beta\Lambda^{-1})^{00}_{ij} = a^0_{ij} - \lambda^{-\delta_{j,1}} a^0_{i,n+1} * \Phi^-_\beta(a^0_{n+1,j})$ is μ times the (i,j)-entry of $A - A\Phi^R_\beta\Lambda^{-1}$. Therefore, there is a well-defined isomorphism $H_0 \to hc'_0(\beta)$ sending a^0_{ij} to μa_{ij} if $i > j$ and a_{ij} otherwise. $\square$

We just showed that HC_0 is infinitely generated for local knots. On the other hand, by Theorem 4.2, HC_0 is always finitely generated for torus knots. Some computer calculations indicate that HC_0 might be finitely generated for non-local knots. This motivates us to come up with following conjecture.

Conjecture 4.1. *Let K be a knot in $S^1 \times S^2$ with framing l, then $HC_0(K; l)$ is finitely generated as an R-algebra if and only if K is not local.*

4.4. *Augmentations*

The presentation for the invariant HC_0 could be very complicated for general knots, especially when the number of crossings is large. It is thus very difficult to analyze the algebraic properties of HC_0 from its presentation. We will deduce a family of invariants, called augmentation numbers, from HC_0. These invariants output a family of integers and could be calculated by computers. The concept of augmentation numbers are introduced in [4, 7] for basically the same reason.

Let $d \geq 2$ be an integer and let $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$. Pick three invertible numbers $\lambda_0, \mu_0, \Gamma_0 \in \mathbb{Z}_d$. Then $\mathbb{Z}_d$ can be treated as an R-module, with λ, μ, Γ acting by multiplication by $\lambda_0, \mu_0, \Gamma_0$, respectively. Then $H(\beta; f; d; \lambda_0, \mu_0, \Gamma_0) := HC_0(\beta; f) \otimes_R \mathbb{Z}_d$ is a $\mathbb{Z}_d$-algebra. Assume $HC_0(\beta; f)$ is finitely generated, then

[d] In [9], a'_{ij} was defined to be μa_{ij} if $i < j$, a_{ij} if $i > j$, and $-1 - \mu$ if $i = j$. After replacing μ by μ^{-1} in the definition of a'_{ij}, $\mu a'_{ij}$ is seen to coincide with a^0_{ij}. Also note that the matrix A in that paper has entries a'_{ij}, but not a_{ij}.

$H(\beta; f; d; \lambda_0, \mu_0, \Gamma_0)$ is a finitely generated $\mathbb{Z}_d$-algebra, and thus has finitely many algebra morphisms into $\mathbb{Z}_d$.

Definition 4.5. Let $\beta \in C_n$, $f \in \mathbb{Z}$, $2 \leq d \in \mathbb{Z}$ such that $HC_0(\beta; f)$ is finitely generated as an R-algebra, and let $\lambda_0, \mu_0, \Gamma_0 \in \mathbb{Z}_d$ be invertible, then $\mathrm{Aug}(\beta; f; d; \lambda_0, \mu_0, \Gamma_0)$ is defined to be the number of algebra morphisms from $H(\beta; f; d; \lambda_0, \mu_0, \Gamma_0)$ to $\mathbb{Z}_d$.

For example, denote the braid $(\alpha_0 \cdots \alpha_{p-1})^q$ representing the (p, q)-torus knot by $T(p, q)$, then $\mathrm{Aug}(T(1, 4); 0; 3; 1, 1, 2) = 4$, $\mathrm{Aug}(T(1, 5); 0; 3; 1, 1, 2) = 2$, $\mathrm{Aug}(T(1, 6); 0; 3; 1, 1, 2) = 4$, $\mathrm{Aug}(T(1, 4); 0; 5; 1, 1, 3) = 6$, $\mathrm{Aug}(T(1, 5); 0; 5; 1, 1, 3) = 3$.

Proposition 4.6. *Set $\lambda = \mu = 1$, then for any $\beta \in \mathcal{C}_n$, there is a $\mathbb{Z}[\Gamma^{\pm 1}]$-algebra morphism from $HC_0(\beta; f)$ to $\mathbb{Z}[\Gamma^{\pm 1}]$ sending each a^x_{ij} to 2Γ.*

Proof. Define $t : \mathcal{A}_n \to \mathbb{Z}[\Gamma^{\pm 1}], t(a^x_{ij}) = 2\Gamma$. We first show for $\beta \in \mathcal{C}_n$, $t\Phi_\beta = t$. It suffices to prove $t\Phi_{\alpha_k} = t, 0 \leq k \leq n-1$. This can be checked directly from Eqs. (2.2) and (2.3).

Similarly, one can prove $t\Phi^+_\beta = t\Phi^-_\beta = t$.

We need to show t factors through $\mathcal{I}_{\beta;f;1,1}$. Note that now $\Lambda_{\beta;f;1,1}$ is the identity matrix.

$t(A^{xy}_{ij} - (\Phi^{-L}_\beta A)^{xy}_{ij}) = 2\Gamma - t(\Phi^-_\beta(a^x_{i,n+1}) * a^y_{n+1,j}) = 2\Gamma - t\Phi^-_\beta(a^x_{i,n+1}) = 2\Gamma - t(a^x_{i,n+1}) = 0$.

In the same way, one can show t factors through the other three relations, and thus t induces an algebra morphism from $HC_0(\beta; f)$ to $\mathbb{Z}[\Gamma^{\pm 1}]$ mapping a^x_{ij} to 2Γ. □

Corollary 4.7. *Let $\beta \in \mathcal{C}_n, f \in \mathbb{Z}$ let $\Gamma_0 \in \mathbb{Z}_d$ be invertible, then* $\mathrm{Aug}(\beta; f; d; 1, 1, \Gamma_0) \geq 1$.

Proof. The map t defined in Proposition 4.6 naturally induces a map from $H(\beta; f; d; 1, 1, \Gamma_0)$ to $\mathbb{Z}_d$. □

5. A Topological Interpretation of the Knot Invariant

In this section, we show that the framed knot invariant HC_0 actually has a rather simple interpretation as the framed cord algebra given in [9, Definition 2.2]. The framed cord algebra is defined for an oriented framed knot K in an oriented 3-manifold M. In the same paper, the author also gave a cord interpretation of the framed cord algebra for knots in S^3 with 0 framing. In the following, we modify the cord interpretation so that it adapts to knots with any framing in any oriented 3-manifold, and prove that the modified version is equivalent to the framed cord algebra. Then we show that the knot invariant HC_0 coincides with the framed cord algebra.

Definition 5.1. Suppose M is an oriented 3-manifold and K is an oriented framed knot in M. Pick a base point $z_0 \in M \backslash K$ for the fundamental group of $M \backslash K$. Let l be the homotopy class of the longitude, determined by the framing of K, in $\pi_1(M \backslash K, z_0)$. Choose a representative curve in the homotopy class l. By abusing the notation, we still denote the representative by l. Fix a point $*$ on l.

(1) A cord in M relative to (K, l) is a continuous map $\gamma : [0,1] \to M \backslash K$, such that $\gamma(0), \gamma(1) \in l$ and $\gamma^{-1}(*) = \emptyset$. Two cords γ_1, γ_2 are said to be equivalent if they are homotopic relative to $l \backslash \{*\}$. Informally speaking, one can slide a cord γ along l, so long as not to pass through the point $*$.

(2) The framed cord algebra, $A(K, l; M)$, is defined as the algebra over R freely generated by the equivalence classes of cords, modulo the ideal generated by the relations given in Fig. 14.

In Fig. 14, the dashed line stands for the curve representing l, and the cord is represented by the solid line transversal to l while the knot is drawn as the solid line parallel to l. The algebra $A(K, l; M)$ is independent of the choice of the base point z_0, the representative curve of the longitude l, and the fixed point $*$.

Now we prove that the framed cord algebra is isomorphic to the one defined in [9]. For the readers' convenience, we first recall the definition of framed cord algebra there.

Definition 5.2 ([9]). Let $K \subset M$ be an oriented framed knot in an oriented 3-manifold M. Pick a base point $z_0 \in M \backslash K$ for the fundamental group of $M \backslash K$.

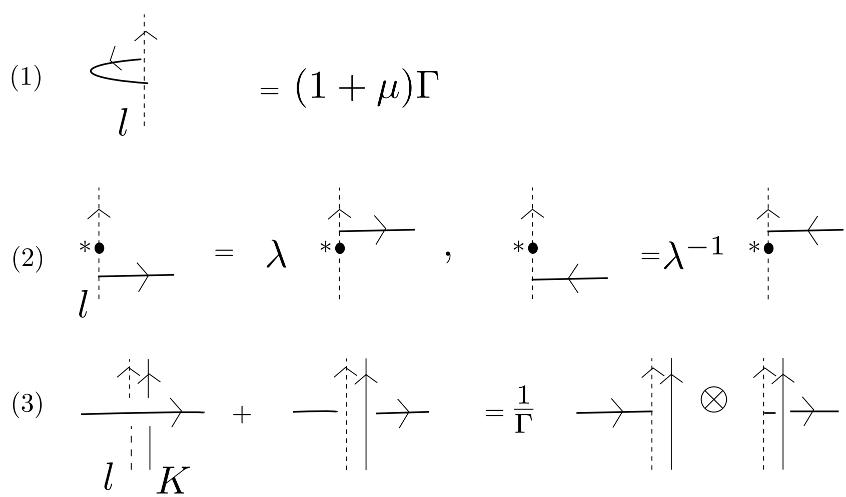


Fig. 14. Skein relations $A(K, l; M)$.

Let l, m denote the homotopy classes of the longitude and meridian of K in $\pi_1(M\backslash K, z_0)$. The framed cord algebra, $\tilde{A}(K, l; M)$, of K is the algebra over R freely generated by the elements of $\pi_1(M\backslash K)$, modulo the ideal generated by the following relations:

(1) $[e] = (1+\mu)\Gamma$;
(2) $[\gamma l] = [l\gamma] = \lambda[\gamma]$ for $\gamma \in \pi_1(M\backslash K, z_0)$;
(3) $[\gamma_1\gamma_2] + [\gamma_1 m \gamma_2] = \frac{1}{\Gamma}[\gamma_1][\gamma_2]$, for $\gamma_1, \gamma_2 \in \pi_1(M\backslash K, z_0)$,

where for an element $\gamma \in \pi_1(M\backslash K, z_0)$, $[\gamma]$ means the image of γ in $\tilde{A}(K, l; M)$.

Remark 5.3. (1) $\tilde{A}(K, l; M)$ does not depend on the choice of the base point z_0 in defining $\pi_1(M\backslash K)$.

(2) The meridian m is oriented as the boundary of a meridian disk, which is oriented so that it, with the orientation on K, is positive.

(3) If we set $\gamma_1 = \gamma, \gamma_2 = e$, then from the first and the third relation, we can derive the relation $[\gamma m] = \mu[\gamma]$. Similarly, we have $[m\gamma] = \mu[\gamma]$.

(4) If $l' = lm^f$ in $\pi_1(M\backslash K, z_0)$, then $\tilde{A}(K, l'; M)$ can be obtained from $\tilde{A}(K, l; M)$ by replacing λ by $\lambda\mu^{-f}$.

Theorem 5.4. *The framed cord algebras defined in Definitions* 5.1 *and* 5.2 *coincide, namely,* $A(K, l; M) \simeq \tilde{A}(K, l; M)$ *for an oriented knot K with framing (longitude) given by l in the manifold M.*

Proof. Assume the base point z_0 is on the curve l, different from the point $*$. For a point $z \in l$, let τ_z be the sub-arc of l connecting z_0 to z not passing the point $*$. Then an element of $\pi_1(M\backslash K, z_0)$ is automatically an equivalence class of cords. Moreover, the three relations in defining $\tilde{A}(K, l; M)$ turn into the three relations defining $A(K, l; M)$, respectively. Conversely, for a cord γ, let $\tilde{\gamma} = \tau_{\gamma(0)} * \gamma * \bar{\tau}_{\gamma(1)}$. Then $\tilde{\gamma}$ is an element of $\pi_1(M\backslash K, z_0)$, and this map also preserves the defining relations of $A(K, l; M)$. It is direct to check these two maps defined above are inverse to each other. ◻

Theorem 5.5. *Let $\beta \in \mathcal{C}_n$ be a braid whose closure is a knot in $S^1 \times S^2$, and let l, m be the homotopy classes of the longitude and the meridian of $\hat{\beta}$, such that $l = [\hat{\beta}']m^f$, where $\hat{\beta}'$ is a parallel copy diagram of $\hat{\beta}$, and $f \in \mathbb{Z}$ is an integer. Then we have $HC_0(\hat{\beta}; l) \simeq A(\hat{\beta}, l; S^1 \times S^2)$.*

Proof. We have $HC_0(\hat{\beta}; l) = HC_0(\beta; f)$ by definition. By Remark 5.3(4) and the definition of $HC_0(\beta; f)$, it suffices to prove the theorem for $f = 0$, namely $l = [\hat{\beta}']$. Set $\Lambda = \Lambda_{\beta;0;1,1}$.

Let $X = D_n \times [0,1]/\{(x,0) \sim (x,1), x \in D_n\}$. Present β as a braid diagram inside X, and assume β intersects D_n in $p_1, \ldots, p_n$. Take a parallel copy diagram β' of β, such that β' intersects D_n in the points $q_1, \ldots, q_n$. Choose some point on $\hat{\beta}'$ right above q_1 as the point $*$. Also we pick two points q_0, q_{n+1} such that q_0 is

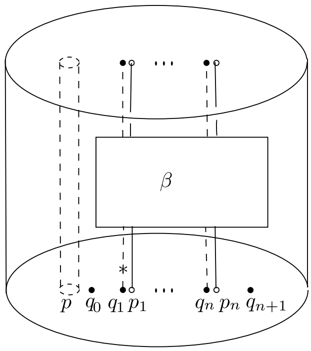


Fig. 15. A picture of X.

near the central puncture p and q_{n+1} is on the right of the puncture p_n near the boundary of D_n. See Fig. 15.

Any cord in $S^1 \times S^2$ relative to $(\hat{\beta}, \hat{\beta}')$ can be homotoped so that it sits inside X. See Remark 2.4 for the relation between X and $S^1 \times S^2$. Then we slide the cord γ along $\hat{\beta}'$ and whenever the cord passes the point $*$, we will multiply by λ or λ^{-1} according to the second relation in Fig. 14. Finally the cord is slide into $D_n \times \{0\}$. We denote the resulting curve by $\tilde{\gamma}$, which is an element in $\mathcal{Q}_n$.

Define the map $\varphi : A(\hat{\beta}, l; S^1 \times S^2) \to HC_0(\beta; 0)$ by sending any cord γ to $\lambda^s \tilde{\gamma}$, where λ^s is the scalar gathered on the way to transit γ into $\tilde{\gamma}$, as stated in the above paragraph. There are several points where we need to check the map is well-defined.

Step 1: The projection of γ to $D_n \times \{0\}$ is not unique, and different projections differ by actions of Φ_β. So we need to show for any $\tilde{\gamma} \in \mathcal{Q}_n$ from q_i to q_j, we have $\tilde{\gamma} = \lambda^{\delta_{i,1}} \Phi_\beta(\tilde{\gamma}) \lambda^{-\delta_{j,1}}$ in $HC_0(\beta; 0)$. Since $\tilde{\gamma}$ can be written as a sum of monomials of the form $a^{x_1}_{i,i_1} a^{x_2}_{i_1,i_2} \cdots a^{x_k}_{i_{k-1},j}$, by Corollary 3.14, $\tilde{\gamma} - \lambda^{\delta_{i,1}} \Phi_\beta(\tilde{\gamma}) \lambda^{-\delta_{j,1}}$ is contained in $\mathcal{I}_{\beta;0;1,1}$ and thus 0 in $HC_0(\beta; 0)$.

Step 2: In $S^1 \times S^2$, the cords have more flexibilities to be homotoped than in X. Precisely, there are two more types of flexibilities. Let γ_1, γ_2 be two curves in D_n such that $\gamma_1(1) = \gamma_2(0) = q_{n+1}, \gamma_1(0) = q_i, \gamma_2(1) = q_j$, for some $1 \le i, j \le n$, and let δ be the loop $\{q_{n+1}\} \times S^1$, then $\gamma_1 * \gamma_2, \gamma_1 * \delta * \gamma_2$ are equivalent cords in $S^1 \times S^2$ but not in X. If we project $\gamma_1 * \delta * \gamma_2$ to $D_n \times \{0\}$, then we get $\lambda^{\delta_{i,1}} \Phi^-_\beta(\gamma_1) * \gamma_2$ or $\gamma_1 * \Phi^-_\beta(\gamma_2) \lambda^{-\delta_{j,1}}$. These are guaranteed by the relations $A - \Lambda \Phi^{-L}_\beta A, A - A\Phi^{-R}_\beta \Lambda^{-1}$. See Remark 3.4(2).

Similarly, in the above argument, if we replace "q_{n+1}" by "q_0", then we get the relations $A - \Lambda \Phi^{+L}_\beta A, A - A\Phi^{+R}_\beta \Lambda^{-1}$.

Step 3: The first and the third relation in Fig. 14 that define $A(\hat{\beta}, l; S^1 \times S^2)$ are apparently mapped to the two "skein" relations in Fig. 5 that define $\tilde{A}_n$. Let γ_1, γ_2 be the two cords shown in Part (1) of the second relation in Fig. 14 (γ_1 being the one on the left-hand side). To compute $\varphi(\gamma_1)$, we can first slide γ_1 through the point $*$ to match γ_2, then project γ_2 to $D_n \times \{0\}$, thus by the design of φ we have $\varphi(\gamma_1) = \lambda\varphi(\gamma_2)$. So φ preserves Part (1) of the second relation. In the same way one can show φ also preserves Part (2) of the second relation.

The above three steps showed that φ is well-defined.

Since elements of $\mathcal{Q}_n$ can each be considered as a cord, we can define a map $\theta : HC_0(\beta; 0) \to A(\hat{\beta}, l; S^1 \times S^2)$ such that $\theta(\gamma_{ij}^x) = \gamma_{ij}^x$. One can check that θ is a well-defined morphism.

Because any cord can be slide into $D_n \times \{0\}$, by an argument similar to the proof of Proposition 2.6, the cord algebra $A(\hat{\beta}, l; S^1 \times S^2)$ is generated by the cords γ_{ij}^x's. Since $\theta\varphi(\gamma_{ij}^x) = \gamma_{ij}^x$, we have $\theta\varphi = Id$ on the algebra $A(\hat{\beta}, l; S^1 \times S^2)$. That $\varphi\theta = Id$ follows directly from the definitions of φ and θ. Therefore, φ is an isomorphism.

□

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