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Fast Track Communication

On enriching the Levin–Wen model with symmetry

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Abstract

Symmetry protected (SPT) and symmetry enriched topological (SET) phases of matter are of great interest in condensed matter physics due to new materials such as topological insulators. The Levin–Wen (LW) model for spin/boson systems is an important rigorously solvable model for studying 2D topological phases. The input data for the LW model is a unitary fusion category, but the same model also works for unitary multi-fusion categories. In this paper, we provide the details for this extension of the LW model, and show that the extended LW model is a natural playground for the theoretical study of SPT and SET phases of matter.

Keywords: Levin–Wen model, multi-fusion category, topological phase

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1. Introduction

Symmetry protected (SPT) and symmetry enriched topological (SET) phases of matter are of great interest in condensed matter physics due to new materials such as topological insulators (see [BBCW, CGLW] and references therein). The Levin–Wen (LW) model for spin/boson systems is an important rigorously solvable model for studying 2D topological phases [LW]. The required input data for the LW model is a unitary fusion category (UFC), but the same model works for unitary multi-fusion categories. In this paper, we provide several results for this extension of the LW model, and show that the extended LW model is a natural playground for the theoretical study of symmetry protected and symmetry enriched topological phases of matter in two spatial dimensions.

The LW model is a Hamiltonian formulation of Turaev–Viro (2 + 1)-TQFTs. Three mathematical theorems underlie this beautiful model: (1) given a UFC $\mathcal{C}$, we can construct a Turaev–Viro unitary (2 + 1)-TQFT [BW], (2) the Drinfeld center $\mathcal{Z}(\mathcal{C})$ or quantum double $D(\mathcal{C})$ of a UFC $\mathcal{C}$ is always modular [Mü], and (3) the Turaev–Viro (2 + 1)-TQFT based on $\mathcal{C}$ is equivalent to the Reshetikhin–Turaev (2 + 1)-TQFT based on the center $\mathcal{Z}(\mathcal{C})$ [BK, TV]. The algebraic model of anyons in the LW model with input $\mathcal{C}$ is encoded by the modular category $\mathcal{Z}(\mathcal{C})$.

We conjecture that all three theorems above have appropriate extensions to unitary multi-fusion categories. Indeed the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of an indecomposable multi-fusion category $\mathcal{C}$ is modular, and a direct sum of modular categories if $\mathcal{C}$ is decomposable. Thus, we expect the Hilbert space $V(S^2)$ of the two-sphere S^2 associated to a decomposable multi-fusion category $\mathcal{C}$ has dimension >1.

There are several generalizations of the LW model, including to 3D and fermion systems [GWW, WW]. The first appearance of a LW model using a unitary multi-fusion category as input is in example H of section III in [LWYW]. While the extension of the LW model to unitary multi-fusion categories as input is straightforward, the application of this extension to symmetry protected and SPT phases of matter is new.

In 2D, the anyon model of a topological phase of quantum matter is algebraically modeled by a unitary modular category $\mathcal{B}$. An exciting new direction is the interplay between symmetry and topological order [BBCW]. But a microscopic physical theory based on local Hamiltonians is still lacking. For topological phases such that $\mathcal{B}$ is a quantum double $\mathcal{B} = D(\mathcal{C})$, the LW model could provide such a microscopic theory. Specifically, given an input $\mathcal{C}$ for the LW model, if the symmetry G could be realized as unitary on-site symmetries of the LW Hamiltonians, then the topological symmetry on $D(\mathcal{C})$ should emerge from the G symmetry of the Hamiltonians. But even for the electric–magnetic duality $e \leftrightarrow m$ of the toric code, a Hamiltonian realization is not in the literature[10]. Current realizations of the $e \leftrightarrow m$ duality need the dual lattice and lattice translation.

In the case of a multi-fusion category, group symmetries sometimes appear in a natural way. For such a category it is natural to consider labels consisting of two indices. We may then endow the half-labels with a group structure G. The pentagon equations are closely related to G-equivariant three-cocycles. This shows, as we demonstrate below, that the LW Hamiltonians naturally come with a G-symmetry. This leads to an application of the LW model to symmetry protected and SPT phases.

To develop a microscopic theory of symmetry enriched and symmetry protected topological phases of matter, we enlarge the local Hilbert spaces that realize the topological order.

[10] Cheng found an on-site realization of the electric–magnetic duality in the toric code, but the details have not been published.

In the LW model case, the input is a numerical description of a UFC $\mathcal{C}$. To add the extra local degrees of freedom, we embed $\mathcal{C}$ into a unitary multi-fusion category $(\mathcal{C}_{ij})_{1\leqslant i,j\leqslant n}$ for some n as a diagonal sub-category $\mathcal{C}_{ii}$ for some $1 \leqslant i \leqslant n$. Since any finite group G is a subgroup of a permutation group S_n for some n, we will seek a Hamiltonian which is invariant under the action G via S_n on the half index set $(i)_{1\leqslant i\leqslant n}$ and/or $(j)_{1\leqslant j\leqslant n}$. Then, when the tetrahedral symbols of $\mathcal{C}$ (which are closely related to the $6j$ symbols) possess some desired symmetries, the Hamiltonian based on the unitary multi-fusion category $(\mathcal{C}_{ij})_{1\leqslant i,j\leqslant n}$ will be G-invariant, and thus realizes a G-SPT order encoded by $D(\mathcal{C})$. The most serious constraint on this procedure is finding tetrahedral symbols with the desired symmetries as solving for $6j$ symbols without any symmetry requirement is already a difficult computational problem. In this paper we obtain some symmetry protected topological phases by implementing this strategy for the simplest nontrivial case: $n = 2$ with $\mathcal{C}$ the trivial UFC.

One potential application of this work is to connect our models to more physically realistic Hamiltonians, such as the $S = \frac{1}{2}$ anti-ferromagnetic Heisenberg $J_1 - J_2$-model on the kagome lattice. The kagome Heisenberg $J_1 - J_2$-model has an $SU(2)$ symmetry and realizes the $D(\mathbf{Z}_2)$ topological order for certain values of J_1, J_2 [JWB]. For any finite subgroup G of $SU(2)$, the kagome Heisenberg model is G-invariant. Since the kagome Heisenberg $J_1 - J_2$ -model and the LW model with $\mathbf{Z}_2$ input are in the same universality class, how to deform one into the other is an open problem of great interest.

The contents of the paper are as follows: in section 2, we provide some background material on multi-fusion categories. In section 3, we give the detail of the extension of the LW model to multi-fusion category inputs and prove that the extended LW models with input $\mathcal{M}_n$ all realize the trivial $(2 + 1)$-TQFT. In section 4, we introduce group structures onto the half-label set of a multi-fusion category and use such group structures to enrich the LW model with symmetries. Finally, we de-equivariantize our G-symmetric LW models with a non-local transformation that leads to traditional LW models coupled with a local group action.

2. Multi-fusion categories and their doubles

All multi-fusion and modular categories in this paper are unitary over the complex numbers $\mathbb{C}$.

2.1. Multi-fusion category

The tensor unit is required to be a simple object in a fusion category. If we allow the tensor unit to be not necessarily simple, we obtain multi-fusion categories. Therefore, a multi-fusion category is a finite semi-simple rigid monoidal $\mathbb{C}$-*linear* category. They arise naturally in mathematics and physics. For example, given a finite depth type $\mathit{\Pi}_1$ sub-factor $N \subset M$ in the study of von Neumann algebras, the $N - N$, $N - M$, $M - N$, and $M - M$ bi-modules form a Morita context, and can be regarded as a multi-fusion category. Much of the fusion category theory naturally generalizes to the multi-fusion case.

Given a multi-fusion category $\mathcal{C}$ with a tensor unit $\mathbf{1}$, the tensor unit $\mathbf{1}$ decomposes into the sum of simple objects $\mathbf{1} \cong \oplus_{i=1}^{n} \mathbf{1}_i$ for some n. For a simple object X of $\mathcal{C}$, there exists a unique pair $1 \leqslant i, j \leqslant n$ such that $\mathbf{1}_i \otimes X \cong X \cong X \otimes \mathbf{1}_j$. Given this pair, we say that X is in the (i, j)th component of $\mathcal{C}$. Let $\mathcal{C}_{ij}$ be the abelian[11] sub-category of $\mathcal{C}$ generated by direct sums of all simple objects in the (i, j)th component. We will call $\mathcal{C}_{ij}$ the (i, j)th component of $\mathcal{C}$. The

[11] Here we mean 'abelian' as in the sense it is used in category theory and homology theory, not as in abelian Anyons.

diagonal components $\mathcal{C}_{ii}$ are fusion categories and the off-diagonal components $\mathcal{C}_{ij}$, $i \neq j$, are $\mathcal{C}_{ii}$-$\mathcal{C}_{jj}$-bimodules. We will call such a multi-fusion category an $n \times n$ multi-fusion category. A 1×1 multi-fusion category is just a fusion category. A multi-fusion category is indecomposable if it is not the direct sum of two non-zero multi-fusion categories.

Definition 2.1. An $n \times n$ two-matrix is an $n \times n$ multi-fusion category for which each component $\mathcal{C}_{i,j}$ is equivalent to $\mathcal{V}ec$, and the fusion rule is $E_{ij} \otimes E_{kl} = \delta_{jk} E_{il}$, where $\{E_{ij}\}_{1 \leqslant i,j \leqslant n}$ is a complete set of isomorphism classes of all simple objects. We will call $\{i\}_{1 \leqslant i \leqslant n}$ the half-label set.

Example 2.2. The $n \times n$ two-matrix $\mathcal{M}_n$.

The multi-fusion category $\mathcal{M}_n$ is the semi-simple category with simple objects $\{E_{ij}\}$, $1 \leqslant i, j \leqslant n$, and fusion rule $E_{ij} \otimes E_{kl} = \delta_{jk} E_{il}$. The tensor product is given by matrix multiplication, which is strictly associative, and the tensor unit is $\mathbf{1} = \oplus_{i=1}^{n} E_{ii}$. $\mathcal{M}_n$ can be regarded as a categorification of the matrix algebra M_n by replacing $\mathbb{C}$ with $\mathcal{V}ec$.

A general object in $\mathcal{M}_n$ is of the form $X = \oplus_{i,j=1}^{n} x_{ij} E_{ij}$, $x_{ij} \in \mathbb{N}$. The multiplicities x_{ij} will be assembled into an $n \times n$ matrix, denoted also as X. So an object X is given by an $n \times n$ matrix $X = (x_{ij})_{1 \leqslant i,j \leqslant n}$ with non-negative integral entries, and E_{ij} is represented by the matrix as the notation indicates: all entries are zero except the (i, j)-entry, which is 1. Then the tensor product of two objects X, Y is just the matrix multiplication XY. For $X = (x_{ij})$, $Y = (y_{ij})$, a morphism from X to Y is of the form $f = (f_{ij})$, where $f_{ij}: x_{ij} E_{ij} \longrightarrow y_{ij} E_{ij}$ can be represented by a linear map from $\mathbb{C}^{x_{ij}} \longrightarrow \mathbb{C}^{y_{ij}}$, or simply a $y_{ij} \times x_{ij}$ matrix. Hence, a morphism in $\mathcal{M}_n$ is simply a matrix of matrices. Then compositions of morphisms are given by entry-wise matrix multiplication.

Example 2.3. Morita contexts as multi-fusion categories.

Suppose $\mathcal{C}$ is a fusion category and $\mathcal{M}$ an indecomposable module category over $\mathcal{C}$. Let $\mathcal{C}_{\mathcal{M}}^{*} = Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ be the dual of $\mathcal{C}$ with respect to $\mathcal{M}$. Then $\begin{pmatrix} \mathcal{C} & \mathcal{M}^{*} \\ \mathcal{M} & \mathcal{C}_{\mathcal{M}}^{*} \end{pmatrix}$ is a 2×2 multi-fusion category.

2.1.1. Quantum doubles. Suppose $\mathcal{C}$ is a multi-fusion category, then its quantum double $D(\mathcal{C})$ in physics or Drinfeld center $\mathcal{Z}(\mathcal{C})$ in mathematics is also a multi-fusion category. Note that $D(\mathcal{C}_1 \oplus \mathcal{C}_2) \cong D(\mathcal{C}_1) \oplus D(\mathcal{C}_2)$ for two multi-fusion categories $\mathcal{C}_i$, $i = 1, 2$. Therefore, we will mainly focus on indecomposable multi-fusion categories.

Theorem 2.4. *Let $\mathcal{C} = (\mathcal{C}_{ij})_{1 \leqslant i,j \leqslant n}$ be an $n \times n$ indecomposable multi-fusion category. Then the quantum double $D(\mathcal{C})$ of $\mathcal{C}$ is equivalent to $D(\mathcal{C}_{ii})$ for any $1 \leqslant i \leqslant n$. It follows that all $\mathcal{C}_{ii}$ are categorically Morita equivalent to each other.*

Proof. If $\mathcal{M}$ is an indecomposable module category over an indecomposable multi-fusion category $\mathcal{C}$, then $D(\mathcal{C}) = D(\mathcal{C}_{\mathcal{M}}^{*})$, where $\mathcal{C}_{\mathcal{M}}^{*}$ is the dual of $\mathcal{C}$ with respect to $\mathcal{M}$ (Corollary 3.35 [EO]). For a fixed i, let $\mathcal{M}_i = \oplus_{k=1}^{n} \mathcal{C}_{ik}$. Then $\mathcal{M}_i$ is an indecomposable $\mathcal{C}$-module category. The dual category of $\mathcal{C}$ with respect to $\mathcal{M}_i$ is $\mathcal{C}_{\mathcal{M}_i}^{*} \cong \mathcal{C}_{ii}^{\mathrm{op}}$, where $\mathcal{C}_{ii}^{\mathrm{op}}$ is the opposite category of $\mathcal{C}$. The theorem now follows from $D(\mathcal{C}) \cong D(\mathcal{C}_{\mathcal{M}_i}^{*}) \cong D(\mathcal{C}_{ii}^{\mathrm{op}}) \cong D(\mathcal{C}_{ii})$. □

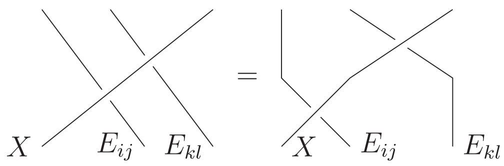


Figure 1. Hexagon equations.

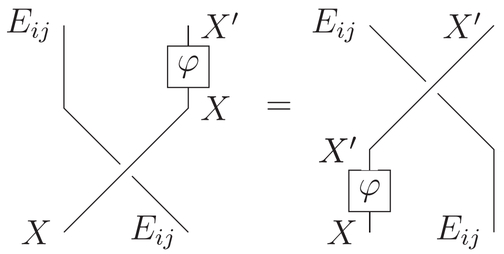


Figure 2. Morphisms in $D(\mathcal{M}_n)$.

2.1.2. Doubles of $n \times n$ two-matrices $\mathcal{M}_n$. It follows from theorem 2.4 that $D(\mathcal{M}_n) \cong \mathcal{V}ec$. To keep our presentation elementary, we provide an explicit proof that $D(\mathcal{M}_n)$ is $\mathcal{V}ec$ in this subsection.

Suppose $X = (x_{ij}) = \oplus\, x_{ij}E_{ij}$ is an object of $\mathcal{M}_n$, and $(X, c_X, \_)$ an object of $D(\mathcal{M}_n)$. Then for any E_{ij}, $c_{X,E_{ij}}$: $X \otimes E_{ij} \longrightarrow E_{ij} \otimes X$ is an isomorphism. Since $X \otimes E_{ij} = \oplus_{k=1}^{n} x_{ki}E_{kj}$, and $E_{ij} \otimes X = \oplus_{k=1}^{n} x_{jk}E_{ik}$, we have $x_{ki} = 0$, $k \neq i$, and $x_{ii} = x_{jj}$ for any pair i, j. Write $x_{ii} = m$, then $X \otimes E_{ij} = mE_{ij} = E_{ij} \otimes X$, and $c_{X,E_{ij}}$ is an $n \times n$ matrix whose (i, j)-entry is an isomorphism $mE_{ij} \longrightarrow mE_{ij}$, i.e. a matrix in $GL(m, \mathbb{C})$, and whose other entries are all 0. Thus an object of $D(\mathcal{M}_n)$ is determined by the set $\{(m, c_{ij})\}$, $1 \leqslant i, j \leqslant n$, where m is a positive integer, and $c_{ij} \in GL(m, \mathbb{C})$. Explicitly, $X = mI_n$, and the half braiding between X and E_{ij} is c_{ij}: $mE_{ij} \longrightarrow mE_{ij}$.

To find the constraints from the hexagon equations as illustrated by figure 1, we see that the left-hand side of the equation in figure 1 is given by $\delta_{jk}c_{il}$: $mE_{il} \longrightarrow mE_{il}$, and the right-hand side is given by $\delta_{jk}c_{jl}c_{ij}$. Thus we obtain

$$c_{ij} = c_{kj}c_{ik},\ \forall\ 1 \leqslant i, j, k \leqslant n. \tag{2.1}$$

Since every c_{ij} is invertible, it follows that $c_{ii} = I_m$, and $c_{ij} = c_{ji}^{-1}$. Hence the c_{ij}'s are completely determined by c_{i1}, $2 \leqslant i \leqslant n$ through the formula $c_{ij} = c_{j1}^{-1}c_{i1}$. The matrices $c_{21}, \cdots, c_{n1} \in GL(m, \mathbb{C})$ can be chosen arbitrarily, and $c_{11} = I_m$. Thus, an object of $D(\mathcal{M}_n)$ is determined by a positive integer m and $(n - 1)$ matrices $c_{21}, \cdots, c_{n1} \in GL(m, \mathbb{C})$.

To understand the morphisms in the doubles, we consider two objects (X, c_{ij}), (X', c'_{ij}), where $X = mI_n$, $X' = m'I_n$. Then a morphism φ: $(X, c_{ij}) \to (X', c'_{ij})$ is given by $(\delta_{ij}\varphi_{ii})$, where φ_{ii}: $mE_{ii} \longrightarrow m'E_{ii}$ is a linear map. This morphism should commute with the half braiding, shown in figure 2.

Figure 2 leads to the following equations for the morphism φ to satisfy:

$$\varphi_{jj}c_{ij} = c'_{ij}\varphi_{ii}.$$

Now assume $m = m'$, and φ_{ii} is an isomorphism. The equations above can be rewritten as $c'_{ij} = \varphi_{jj} c_{ij} \varphi_{ii}^{-1}$. By equation (2.1), it suffices to satisfy $c'_{i1} = \varphi_{11} c_{i1} \varphi_{ii}^{-1}$ for $i = 2, \cdots n$. Using the freedom for choosing φ_{ii}, we choose them so that $c'_{i1} = I_m$ for all i, and thus $c'_{ij} = I_m$, $\forall 1 \leqslant i, j \leqslant n$. Therefore, two objects of $D(\mathcal{M}_n)$ are isomorphic if and only if their diagonal entries m and m' are the same, i.e. an isomorphism class is uniquely determined by a positive integer m. For each m, we choose a representative $(X, c_{ij}) = (mI_n, I_m)$, which is denoted as (m).

Note that $(m) \oplus (m') = (m + m')$. Hence, $D(\mathcal{M}_n)$ is generated by the single object $(1) = (I_n, 1)$. Note that $Hom((1), (1)) = \mathbb{C}$, so (1) is the only simple object in the category. Thus, $D(\mathcal{M}_n) = \mathcal{V}ec$ as expected.

3. LW model for multi-fusion categories

Fix an integer $d \geqslant 2$, and a cellulation γ of an oriented closed surface Y. We often also refer to γ as a graph in Y by thinking about the one-skeleton of γ. Let $V(\gamma)$, $E(\gamma)$, and $F(\gamma)$ be the set of vertices (sites), edges (bonds), and faces (plaquettes) of γ, respectively. Then $L_\gamma(Y)$ will be the local Hilbert space $\otimes_{e \in E(\gamma)} \mathbb{C}^d$, i.e. we attach a qudit $\mathbb{C}^d$ to each edge. The orthonormal basis of $L_\gamma(Y)$ consists of all colors of the edges by a basis of $\mathbb{C}^d$. In this section, d will be the rank of the input UFC $\mathcal{C}$, i.e., the number of labels.

Definition 3.1. A Hamiltonian H is a commuting local projector (CLP) Hamiltonian if $H = \sum_\alpha P_\alpha$, where P_α is a collection of pair-wise commuting local orthogonal projectors.

In general, we are not really interested in a single CLP Hamiltonian, rather a prescription for writing down a family of CLP Hamiltonians on all local Hilbert spaces $L_\gamma(Y)$ associated to cellulations γ of Y. Such a prescription will be called a *Hamiltonian schema*. Since we are interested in thermodynamical physics, we need to study limits when the size of cellulations measured by the mesh goes to 0. We can use Pachner's theorem to organize all triangulations of a surface into a directed set. Then local Hilbert spaces and their ground state manifolds form inverse systems of finite dimensional Hilbert spaces.

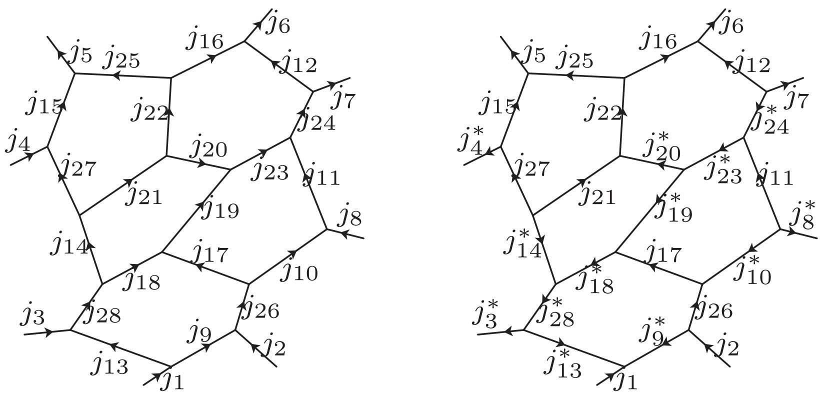
The numerical data to specify the local Hilbert space and Hamiltonian of a LW model is a description of a UFC in terms of $6j$-symbols. In order to implement unitarity and symmetries, we demand some symmetries of the $6j$ symbols [Wan]. There are subtleties when the input UFC has multiplicities in the fusion rules, as defined below, and non-trivial Frobenius–Schur indicators. In the following, we will assume that all UFCs are multiplicity free and their modified $6j$-symbols, called tetrahedral symbols, have the full tetrahedral symmetry, as defined below. Not all UFCs have tetrahedral symbols that have the full tetrahedra symmetry [Ho].

3.1. LW Hamiltonian schema for unitary fusion categories

A *label set* L is a finite set with a distinguished element 0 and with an involution $^*: L \to L$ such that $0^* = 0$. Elements of L are called labels, 0 is called the trivial label, and $j^* \in L$ is called the dual of $j \in L$.

A fusion rule on L is a map $N: L \times L \times L \to \mathbb{N}$ such that for $a, b, c, d \in L$

$$N_{0a}^{b} = N_{a0}^{b} = \delta_{ab}, \tag{3.1}$$



(a) (b)

Figure 3. A configuration of string types on a directed trivalent graph. The configuration (b) is treated the same as (a), with some of the directions of some edges reversed and the corresponding labels j conjugated j^*.

$$N^0_{ab} = \delta_{ab^*}, \tag{3.2}$$

$$\sum_{x\in L} N^x_{ab}N^d_{xc} = \sum_{x\in L} N^d_{ax}N^x_{cd}. \tag{3.3}$$

A fusion rule is multiplicity-free if $N^c_{ab} \in \{0, 1\}$ for all $a, b, c \in L$. Set $\delta_{abc} := N^{c^*}_{ab}$, then $\delta_{abc} = \delta_{bca}$ and $\delta_{abc} = \delta_{c^*b^*a^*}$. A triple (a, b, c) is admissible if $\delta_{abc} = 1$.

Given a fusion rule on L, a loop weight is a map $w: L \to \mathbb{R}\backslash\{0\}$ such that $w_{a^*} = w_a$ and

$$\sum_{c\in L} w_c\delta_{abc^*} = w_a w_b. \tag{3.4}$$

In particular, $w_0 = 1$. For unitary modular categories, the quantum dimensions—quantum traces of the identity morphisms—satisfy $d_j \geqslant 1$ for all $j \in L$. Quantum dimensions might differ from loop weights $\{w_i\}$. We let $\alpha_i = \frac{d_i}{w_i} = \pm 1$ for each label, and require:

$$\alpha_i\alpha_j\alpha_k = 1, \quad \text{if } \delta_{ijk} = 1. \tag{3.5}$$

A symmetrized tetrahedral symbol is a map $T: L^6 \to \mathbb{C}$ satisfying the following conditions:

$$\text{tetrahedral symmetry: } T^{ijm}_{kln} = T^{mij}_{nk^*l^*} = T^{klm^*}_{ijn^*} = \alpha_m\alpha_n\overline{T^{j^*i^*m^*}_{l^*k^*n}}, \tag{3.6}$$

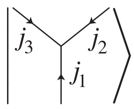
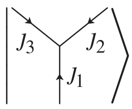
$$\text{pentagon identity: } \sum_n w_n T^{mlq}_{kp^*n} T^{jip}_{mns^*} T^{js^*n}_{lkr^*} = T^{jip}_{q^*kr^*} T^{riq^*}_{mls^*}, \tag{3.7}$$

$$\text{orthogonality condition: } \sum_n w_n T^{mlq}_{kp^*n} T^{l^*m^*i^*}{}_{pk^*n} = \frac{\delta_{iq}}{\mathrm{w}_i}\delta_{mlq}\delta_{k^*ip}. \tag{3.8}$$

For convenience, we consider LW models defined on trivalent graphs in a closed oriented surface. Initially, we choose an arrow of each edge to assign a label, but the Hilbert space does not depend on these arrows, by using the following identification: for any state

$|\psi\rangle \in L_\gamma(Y)$, if we reverse the direction of an edge e and replace its label j_e by its dual j_e^*, then the resulting state is identified with the initial state $|\psi\rangle$. See figure 3.

There are two types of local operators, Q_v which are defined at vertices v and B_p^s which are defined at a plaquette for an $s \in L$. Let us first define the operator Q_v. On a trivalent graph, Q_v acts on the labels of three edges incoming to the vertex v. We define the action of Q_v on the basis vector with j_1, j_2, j_3 by

Q_v  $= \delta_{j_1 j_2 j_3}$  (3.9)

where the tensor $\delta_{j_1 j_2 j_3}$ equals either 1 or 0, which determines whether the triple (j_1, j_2, j_3) is 'allowed' to meet at the vertex. Since $\delta_{j_1 j_2 j_3} = \delta_{j_2 j_3 j_1}$, the ordering in the three labels is not important. To be compatible with the conjugation structure of labels, the branching rule must satisfy $\delta_{0jj^*} = \delta_{0j^*j} = 1$, $\delta_{0ij^*} = 0$ if $i \neq j$, and $\delta_{j_1 j_2 j_3} = \delta_{j_3^* j_2^* j_1^*}$.

One important property of the tetrahedral symbols is that

$$T^{ijm}_{kln} = 0 \quad \text{unless } \delta_{ijm} = \delta_{klm^*} = \delta_{lin} = \delta_{nk^*j^*} = 1. \tag{3.10}$$

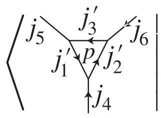
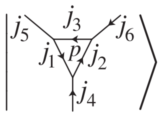
This is a consequence of the orthogonality condition and the tetrahedral symmetry.

For convenience, we take the square root of the loop weight as follows. We define

$$v_j := \frac{1}{T^{j^* j 0}_{0 0 j}}. \tag{3.11}$$

We can verify $v_j^2 = w_j$ from the orthogonality condition. In particular, $v_0 = 1$.

The operator B_p^s acts on the boundary edges of the plaquette p, and has the matrix elements on a triangle plaquette

 B_p^s 

$$= v_{j_1} v_{j_2} v_{j_3} v_{j_1'} v_{j_2'} v_{j_3'} T^{j_5 j_1^* j_3}_{s j_3' j_1'^*} T^{j_4 j_2^* j_1}_{s j_1' j_2'^*} T^{j_6 j_3^* j_2}_{s j_2' j_3'^*}. \tag{3.12}$$

The same rule applies when the plaquette p is a quadrangle, a pentagon, or a hexagon and so on. Note that the matrix is nondiagonal only on the labels of the boundary edges (i.e., j_1, j_2, and j_3 on the above graph).

The operators B_p^s have the properties

$$B_p^{s\dagger} = B_p^{s^*}, \tag{3.13}$$

$$B_p^r B_p^s = \sum_t \delta_{rst^*} B_p^t. \tag{3.14}$$

The Hamiltonian of the model is

$$H = -\sum_v Q_v - \sum_p B_p, \qquad B_p = \frac{1}{D} \sum_s w_s B_p^s, \tag{3.15}$$

where $D = \sum_j d_j^2$, and the sum runs over all vertices v and all plaquettes p of the trivalent graph.

The main property of the interactions Q_v and B_p is that they are mutually commuting, orthogonal projection: (1) $[Q_v, Q_{v'}] = 0 = [B_p, B_{p'}]$, $[Q_v, B_p] = 0$; (2) $Q_v^2 = Q_v = Q_v^*$ and $B_p^2 = B_p = B_p^*$. Thus the Hamiltonian is exactly soluble. The elementary energy eigenstates are given by common eigenvectors of all these projections. The ground states have eigenvalues $Q_v = B_p = 1$ for all v and p, while each excited state violates these constraints for some subset of the plaquettes and vertices.

3.2. Multi-fusion category extension of the LW model

The input data for LW models can be extended to the multi-fusion case. The extension is to replace the trivial label 0 by a subset L_0 of L, in order to numerically specify the (not necessarily simple) tensor unit of the category.

We start with a label set L with an involution $^*: L \to L$ that is equipped with a trivial set L_0, where L_0 is determined by the decomposition of the tensor unit into simple objects as in section 2.1. A fusion rule on L is a map $N: L \times L \times L \to \in$ satisfying that for all $a, b, c, d \in L$

$$\sum_{\alpha \in L_0} N_{\alpha a}^{b} = \sum_{\alpha \in L_0} N_{a\alpha}^{b} = \delta_{ab}, \tag{3.16}$$

$$\sum_{\alpha \in L_0} N_{ab}^{\alpha} = \delta_{ab^*}, \tag{3.17}$$

$$\sum_{x \in L} N_{ab}^{x} N_{xc}^{d} = \sum_{x \in L} N_{ax}^{d} N_{cd}^{x}. \tag{3.18}$$

These three equations are obtained by formally replacing 0 by $\sum_{\alpha \in L_0} \alpha$ in equations (3.1)–(3.3). Since $N_{\alpha a}^{b} \in \mathbb{N}$, the first equality implies that for each label $a \in L$, there exists a unique pair $(\alpha, \beta) \in L_0 \times L_0$ such that $N_{\alpha' a}^{b} = \delta_{ab}\delta_{\alpha'\alpha}$ and $N_{a\beta'}^{b} = \delta_{ab}\delta_{\beta'\beta}$ for $b \in L$, $\alpha', \beta' \in L_0$. We say a has the grading (α, β). Obviously, each $\alpha \in L_0$ has the grading (α, α).

Therefore L is graded by $L_0 \times L_0$: $L = \bigsqcup_{\alpha,\beta \in L_0} {}_{\alpha}L_{\beta}$, and we can denote the labels in ${}_{\alpha}L_{\beta}$ by ${}_{\alpha}a_{\beta}$ to specify their gradings (α, β). Equations (3.16) and (3.18) imply

$$N_{{}_{\alpha}a_{\beta}, {}_{\gamma}b_{\delta}}^{{}_{\epsilon}c_{\zeta}} = 0 \quad \text{unless } \alpha = \epsilon,\ \beta = \gamma,\ \delta = \zeta. \tag{3.19}$$

Together with equation (3.17), it implies

$$\alpha^* = \alpha \quad \text{for } \alpha \in L_0, \tag{3.20}$$

$${}_{\alpha}a_{\beta}^{*} \in_{\beta} L_{\alpha} \quad \text{for } {}_{\alpha}a_{\beta} \in_{\alpha} L_{\beta}. \tag{3.21}$$

Given a fusion rule on $\{L, L_0\}$, the loop weight satisfies

$$\sum_{{}_{\alpha}c_{\gamma} \in {}_{\alpha}L_{\gamma}} \delta_{{}_{\alpha}a_{\beta}, {}_{\beta}b_{\gamma}, ({}_{\alpha}c_{\gamma})^*} \mathrm{w}_{{}_{\alpha}c_{\gamma}} = \mathrm{w}_{{}_{\alpha}a_{\beta}} \mathrm{w}_{{}_{\beta}b_{\gamma}}. \tag{3.22}$$

The symmetrized tetrahedral symbols are defined in the same way as those in the previous section, and so are the LW models. This leads to the following conclusion:

Proposition 3.2. *Using the modified label set L with trivial set L_0, the LW Hamitonian schemas extend to multi-fusion categories, and all resulting Hamiltonians are CLPs.*

3.3. The $n \times n$ two-matrix $\mathcal{M}_n$ as input

Consider the multi-fusion category $\mathcal{M}_n$ from example 2.2. This example gives the following data. The label set is $L = \{E_{ij}\}$, the trivial set is $L_0 = \{E_{ii}\}$, and the fusion rule is

$$\delta_{E_{ij},E_{kl},E_{mn}} = \delta_{jk}\delta_{lm}\delta_{ni}. \tag{3.23}$$

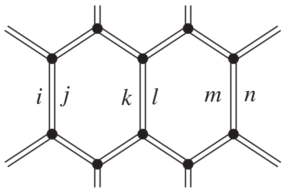
The set $L = \sqcup_{i,j} {}_iL_j$ is graded by i, j where each ${}_iL_j$ has only one element, E_{ij}. The duals are $E_{ij}^* = E_{ji}$.

Let us set the loop weights to be $w_{E_{ij}} = 1$ for all i, j. The simplest normalized $6j$-symbol is to take

$$T_{def}^{abc} = \begin{cases} 1 & \text{if } \delta_{abc} = \delta_{dec^*} = \delta_{eaf} = \delta_{fd^*b^*} = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.24}$$

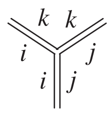
for $a, b, c, d, e, f \in L$.

The local Hilbert space is spanned by labels on all edges. In our example, labels are the gradings (i, j). Graphically, we use a double line to represent the gradings as illustrated below.



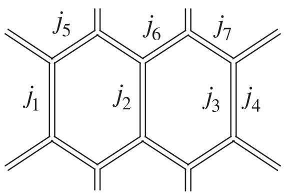
We do not draw arrows in the graph as a label on each arrowed edge is identified with its dual on the same edge with the arrow reversed. For example, the labels on the three vertical edges illustrated above read as E_{ij}, E_{kl} and E_{mn} upwards, and as E_{ji}, E_{lk} and E_{nm} downwards.

Consider the eigenspace $\mathcal{L}^{Q=1}$ of $Q_v = 1$ for all vertices. The fusion rule in equation (3.23) has a double line representation near each vertex of the form



which presents an admissible triple (E_{ij}, E_{jk}, E_{kl}) on the three edges incoming into the vertex, and for which all other combinations are not allowed. If two lines are connected, then they carry the same label i.

Therefore the basis vectors in $\mathcal{L}^{Q=1} = \otimes_p \mathbb{C}^n$ have a double line representation as below.



To each plaquette p, there is a loop labeled by j_p. The basis is denoted in terms of the loop labels j_p and given by $\{|j_1, j_2, ...\rangle\}$. This statement holds for the model on any closed surface.

The operator B_p is now $B_p = \frac{1}{n}\sum_{\alpha\beta} B_p^{E_{\alpha\beta}}$, where $B_p^{E_{\alpha\beta}}$ is defined in equation (3.12). In the subspace $\mathcal{L}^{Q=1}$, $B_p^{E_{\alpha\beta}}$ is a map

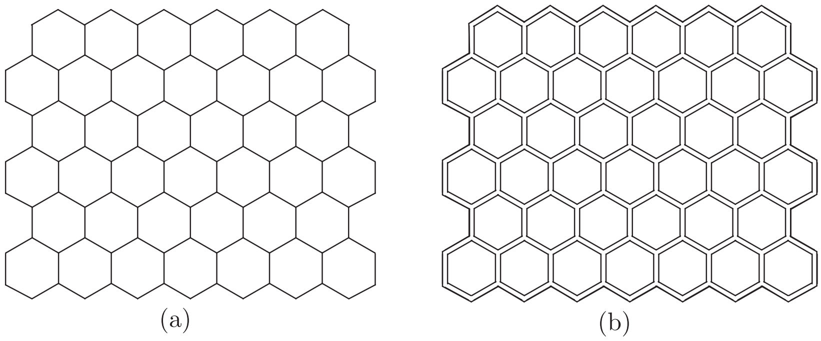


Figure 4. (a) Disk with a loop boundary. (b) Double line representation for $\mathcal{L}^{Q=1}$.

$$B_p^{E_{\alpha\beta}}: \left| j_1, j_2, ..., j_p, ... \right\rangle \mapsto \delta_{\beta, j_p} \left| j_1, j_2, ..., \alpha, ... \right\rangle. \tag{3.25}$$

Therefore there is only one ground state, with common eigenvalues $Q_v = 1$ and $B_p = 1$ for all v, p:

$$|\Phi\rangle = \sum_{\alpha_1, \alpha_2, ...} \left| \alpha_1, \alpha_2, ..., \alpha_p, ... \right\rangle, \tag{3.26}$$

up to a constant normalization factor. The discussion can be summarized by the following proposition.

Proposition 3.3. *The LW Hamiltonian schemas with input $\mathcal{M}_n$ for all $n \geqslant 1$ realize the trivial* $(2 + 1)$*-TQFT.*

Consider now the example $n = 2$, for which it is easy to give an explicit description of the ground state. In this case the operator B_p is the matrix $\frac{1}{2}(\mathbf{1} + \sigma^x)$ in the local basis $|i_p\rangle$, where $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a Pauli matrix. Dropping the constant terms, we can write the Hamiltonian in the subspace $\mathcal{L}^{Q=1}$ as

$$H\,|_{Q=1} = -\frac{1}{2} \sum_p \sigma_p^x. \tag{3.27}$$

It is convenient to use the dual graph picture. Namely, by taking the dual graph of a spatial trivalent graph, we obtain a triangulation of the surface. Then the ground state is simply a tensor product $\otimes_p |\sigma_p^x = 1\rangle$ of all local eigenstates of $\sigma^x = 1$ at the vertices of the dual triangulation.

3.4. Degeneracy on a disk

Consider the disk with a smooth loop boundary. On the graph in figure 4(a), the Hamiltonian takes the form in equation (3.15), with the first summation over all vertices of the graph and over all internal plaquettes inside the disk.

The double line representation for $\mathcal{L}^{Q=1}$ is illustrated in figure 4(b). A basis vector in $\mathcal{L}^{Q=1}$ is denoted by $|\alpha_\partial; \alpha_1, \alpha_2, ..., \alpha_p, ...\rangle$, specified by a loop value α_p associated to each plaquette p inside the disk, and a loop value α_∂ associated to the boundary.

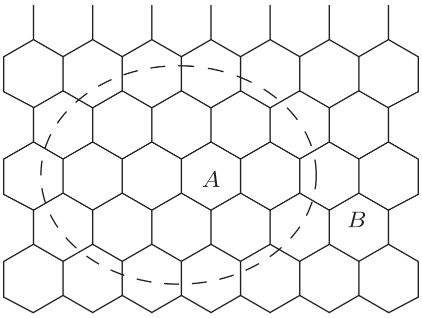


Figure 5. Partition into subsystems A and B with the boundary along a dashed curve.

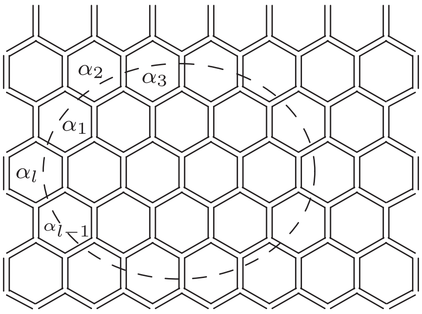


Figure 6. Nonzero contributions to the entanglement spectrum are specified by the loop labels α_1, α_2,..., α_l on the boundary.

The second term $-\sum_p B_p$ in the Hamiltonian does not affect α_∂. Therefore, the ground states are degenerate and paramterized by α_∂. For the input data $\mathcal{M}_n$, the ground state degeneracy is n.

Similar to the formula in equation (3.26), the degenerate ground states for all α_∂ are

$$\left|\Phi(\alpha_\partial)\right\rangle = \sum_{\alpha_1,\alpha_2,\ldots} \left|\alpha_\partial;\ \alpha_1,\ \alpha_2,\ldots,\alpha_p,\ldots\right\rangle. \tag{3.28}$$

3.5. Topological entanglement entropy

Consider the extended LW model with $\mathcal{M}_n$ as input. We divide a trivalent graph into two subsystems A and B, where their boundary intersects some edges, denoted by a dashed curve as illustrated in figure 5.

Denote the edges across the boundary by $j_1, j_2, \ldots, j_l \in L$, or simply $\{j_i\}$ for short. The number l will be called the length of the boundary curve.

The reduced density matrix for the ground state Φ in equation (3.26) is defined by $\rho_A = \oplus_{\{j_i\}} \rho_A^{\{j_i\}}$, where

$$\rho_A^{\{j_i\}} = \mathrm{tr}_B\left[\left\langle \left\{j_i\right\}\right|\left(|\Phi\rangle\langle\Phi|\right)\left|\left\{j_i\right\}\right\rangle\right]. \tag{3.29}$$

Here tr_B is the partial trace over all labels in the subsystem B.

By definition, the entanglement entropy is

$$S_E = -\mathrm{tr}_A\left(\rho_A \log \rho_A\right), \tag{3.30}$$

where we calculate the entanglement entropy on the two-sphere.

The double line representation provides a clear picture of the spectrum of ρ_A: nonzero contributions to the entanglement spectrum are specified by the loop labels $\alpha_1, \alpha_2, ..., \alpha_l$ on the boundary, see figure 6. Specifically, in terms of the new basis of the subspace $\mathcal{L}^{Q=1}$, the boundary is specified by the loop labels $\alpha_1, \alpha_2, ..., \alpha_l$. $\rho_A^{\{j_i\}}$ has exactly one nonzero eigenvalue λ if and only if the boundary configuration $\{j_i\}$ has the following form:

$$\alpha_l \Big|\Big| \alpha_1 \qquad \alpha_1 \Big|\Big| \alpha_2 \qquad \alpha_2 \Big|\Big| \alpha_3 \qquad \cdots \qquad \alpha_{l-1} \Big|\Big| \alpha_l$$

By symmetry, ρ_A has n^l equal eigenvalues, which are normalized to $\lambda = 1/n^l$ by the trace condition $\mathrm{tr}_A(\rho_A) = 1$. It follows that

$$S_E = \log(n) l. \tag{3.31}$$

Since there is not any sub-leading correction term in S_E—it is exactly proportional to the length l of the boundary curve—the topological entanglement entropy is 0 [KP, LW2]. A similar calculation on the torus also leads to zero topological entanglement entropy.

4. Symmetry enriching the LW model

We are interested in enriching the LW model with on-site unitary symmetries. A good example is the toric code Hamiltonian $H = -\sum_v A_v - \sum_p B_p$ on the square lattice, where a qubit is one each edge. As usual, the vertex operator A_v is the tensor product of σ^x and the identity, while the plaquette term is a tensor product of σ^z and the identity. A moment's thought shows that the tensor product of σ^x (or σ^z) over all edges is an on-site unitary symmetry of the toric code Hamiltonian. Of course this $\mathbb{Z}_2$ symmetry is very trivial because it will not permute anyon types. But even if a $\mathbb{Z}_2$ symmetry of the toric code does not permute anyon types, there are still four different ways to fractionalize a $\mathbb{Z}_2$ symmetry in a one-to-one correspondence to classes in $H^2(\mathbb{Z}_2; \mathbb{Z}_2^2) = \mathbb{Z}_2^2$ [BBCW]. In this section, we will describe analogous symmetries of the LW Hamiltonians. It will be interesting to understand their role in a microscopic theory of symmetry fractionalization, symmetry defects, and gauging using fixed-point rigorously solvable Hamiltonians.

4.1. Classification of n × n two-matrices

The half-label set can be endowed with a group structure. In this subsection, we classify all $n \times n$ two-matrices whose half-label set has the structure of an abelian group G.

By the fusion rule, there are four independent variables in the $6j$-symbols. Denote them by

$$\phi_4(\alpha, \beta, \gamma, \delta) := T^{E_{\alpha\beta}E_{\beta\gamma}E_{\gamma\delta}}_{E_{\gamma\delta}E_{\delta\alpha}E_{\beta\delta}} \mathrm{w}_{E_{\beta\delta}}. \tag{4.1}$$

In this notation the pentagon identity can be written as

$$\phi_4(\alpha, \beta, \gamma, \delta)\phi_4(\alpha, \beta, \delta, \epsilon)\phi_4(\beta, \gamma, \delta, \epsilon) = \phi_4(\alpha, \gamma, \delta, \epsilon)\phi_4(\alpha, \beta, \gamma, \epsilon), \tag{4.2}$$

for $\alpha, \beta, \gamma, \delta = 1, 2, ..., n$.

Suppose the half labels $\alpha, \beta, \dots$ form a finite group G with $|G| = n$, e.g. $G = \mathbb{Z}_n$. Recall that a homogeneous n-cochain taking values in $\mathbb{C}$ is a map ϕ_{n+1}: $G^{n+1} \to \mathbb{C}\backslash\{0\}$ such that $g \cdot \phi_{n+1}(g_1, \dots g_{n+1}) = \phi_{n+1}(gg_1, \dots, gg_{n+1})$. We will usually consider the trivial G-action on $\mathbb{C}\backslash\{0\}$. Hence, ϕ_4: $G^4 \to \mathbb{C}\backslash\{0\}$ is a homogeneous three-cochain on G, equipped with an action:

$$g \cdot \phi_4(\alpha, \beta, \gamma, \delta) = \phi_4(g\alpha, g\beta, g\gamma, g\delta), \tag{4.3}$$

where we regard $\mathbb{C}\backslash\{0\}$ as a trivial G-module. The pentagon identity (4.2) can then identified with the three-cocycle condition $\delta\phi_4 = 1$, where the coboundary δ is defined by

$$\delta\phi_4(\alpha_0, \alpha_1, \dots, \alpha_4) = \prod_{0 \leqslant i \leqslant 4} \phi_4(\alpha_0, \alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_4)^{(-1)^i}. \tag{4.4}$$

Therefore, the $6j$-symbols are classified by the third group cohomology classes in $H^3(G, U(1))$. Note that not all three-cocycles satisfy the tetrahedral symmetry in equation (3.6). We call three-cocycles ϕ_4 defined as above G-invariant.

Definition 4.1. Given a finite group G and a homogeneous three-cocycle ϕ_4, ϕ_4 is called G-invariant if $\phi_4(\alpha, \beta, \gamma, \delta) = \phi_4(g\alpha, g\beta, g\gamma, g\delta)$ for all $\alpha, \beta, \gamma, \delta = 1, \cdots, n$, and $g \in G$. That is the action of G on ϕ_4 given by equation (4.3) is trivial if $\mathbb{C}\backslash\{0\}$ is regarded as a trivial G-module.

Consider the case where $n = 2$. Then the group is $\mathbb{Z}_2 = \{0, 1\}$. There are two equivalence classes, with the three-cocycle representatives:

(1) $w_{E_{\alpha\beta}} = 1$, and $\phi_4 = 1$ is constant, as in section 3.3;

(2) $w_{E_{\alpha\beta}} = \begin{cases} 1 & \text{if } \alpha = \beta \\ -1 & \text{if } \alpha \neq \beta \end{cases}$, and

$$\phi_4(\alpha, \beta, \gamma, \delta) = \exp\left[\frac{\pi \mathrm{i}}{2}(2 - |\alpha + \beta + \gamma + \delta - 2|)\right] w_{E_{\beta\delta}}.$$

The two representatives are chosen to satisfy the tetrahedral symmetry in equation (3.6). The G-actions in equation (4.3) on both three-cocycles are trivial, hence both three-cocycles are $\mathbb{Z}_2$-invariant.

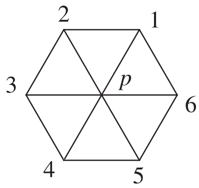
Similar to equation (3.27), the Hamiltonian for the second class can be written as

$$H = -\frac{1}{2}\sum_p \tau_p^x. \tag{4.5}$$

In the dual triangulation, τ^x is

$$\tau^x = \left\{ \prod_{\langle ij \rangle \in \partial p} \exp\left[\mathrm{i}\frac{\pi}{4}\left(\mathbf{1} - \sigma_i^z \sigma_j^z\right) + \mathrm{i}\frac{\pi}{2}\left(\mathbf{1} + \sigma_i^z \sigma_j^z\right)\right]\right\} \sigma_p^x, \tag{4.6}$$

with the product over nearest neighbor vertex pairs on the boundary of p, for example, over $\langle 12 \rangle, \langle 23 \rangle, \dots, \langle 61 \rangle$ in the example below:



Here only the relevant triangles of the dual graph are shown, assuming the remaining part of the graph is not affected.

4.2. G-symmetric Hamiltonian schema

Given a homogeneous three-cocycle ϕ_4, not necessarily G-invariant, we have a multi-fusion category $(\mathcal{M}_n, \phi_4)$ with 6j-symbols given by equation (4.1). This in turn allows us to define a LW Hamiltonian schema with this multi-fusion category as input.

Definition 4.2. Given a finite group G and a LW Hamiltonian schema, the LW Hamiltonian schema is G-symmetric if each $g \in G$ acts on the qudit $\mathbb{C}^d$ as a unitary matrix U_g, such that it is a symmetry of all resulting LW Hamiltonians.

Theorem 4.3. *If the homogeneous three-cocycle ϕ_4 for an $n \times n$ two-matrix is G-invariant, then the LW Hamiltonian schema with the $n \times n$ two-matrix $(\mathcal{M}_n, \phi_4)$ input is G-symmetric, and realizes a G-symmetry protected topological phase (SPT).*

Using proposition 3.3, we just need to check the G-invariance of LW Hamiltonians, which is a straightforward check. But it is not clear if we have realized any non-trivial SPTs, which will be addressed in the next section.

We conjecture that this result can be extended in the following way.

Conjecture 4.4. The LW Hamiltonian schema with an $n \times n$ multi-fusion C input realizes a SPT phase $D(C)$ with some on-site unitary symmetry G, which does not permute anyon types.

4.3. De-equivariantizing the G-symmetric LW model

To understand if the SPTs realized in theorem 4.3 are non-trivial, we study the gauging of the symmetry G [BBCW, LG]. First we give a proof of the following proposition.

Proposition 4.5. *There is a non-local transformation from G-symmetric LW models to traditional LW models coupled to a local action.*

Given a finite group G, a homogeneous three-cocycle ϕ_4 of G can be de-equivariantized to obtain an inhomogeneous three-cocycle φ_3 by setting

$$\varphi_3(x, y, z) = \phi_4(1, x, xy, xyz), \tag{4.7}$$

for $x, y, z \in G$ and 1 is the identity element of G. The three-cocycle φ_3 has a group action

$$g \cdot \varphi_3(x, y, z) = \phi_4(g, gx, gxy, gxyz). \tag{4.8}$$

The inhomogeneous three-cocycles φ_3 and homogeneous three-cocycles ϕ_4 are in one–one correspondence because ϕ_4 can be recovered from φ_3 by

$$\phi_4(\alpha, \beta, \gamma, \delta) = \alpha \cdot \varphi_3\left(\alpha^{-1}\beta, \beta^{-1}\gamma, \gamma^{-1}\delta\right). \tag{4.9}$$

This de-equivariantization reduces the G-symmetric data from a multi-fusion category to input data from an abelian modular category $\mathcal{V}ec_G^{\varphi_3}$ with a nontrivial action of G on φ_3.

The correspondence between ϕ_4 and φ_3 can be adapted to the local Hilbert spaces and their Hamiltonians, therefore, the correspondence establishes a non-local duality transformation. In the following, we will work with the dual triangulations and consider only the two-sphere S^2 for simplicity.

For the local Hilbert spaces, the subspaces $\mathcal{L}^{Q=1}$ are spanned by the group elements $\{\alpha_p\}$ at vertices p of the dual triangulations. Choose an arbitrary vertex p_0, and designate it as the origin.

On the two-sphere, the set of group elements $\{\alpha_0, \alpha_1, \alpha_2, ...\}$ assigned to vertices corresponds to the set of group elements $\{g_1, g_2, ...\}$ assigned to edges satisfying the following condition: around any triangle, the holonomy (the product of the three group elements around the triangle) is equal to the identity 1. In fact, the group element g_e on each edge e can be written as $g_e = \alpha_2\alpha_1^{-1}$, so it is determined by α_1 (α_2) at the starting (ending) point of e. Conversely, given α_0 at the origin vertex p_0, α_p can be determined as follows: choose an arbitrary path from p_0 to p, multiply the group elements on the edges along the path and α_0. The two constructions above give rise to an isomorphism

$$\left\{\alpha_0, \alpha_1, \alpha_2, ...\right\}\Big|_{\text{vertex colors}} \cong \left\{\alpha_0; g_1, g_2, ...\right\}\Big|_{\text{trivial holonomy}}, \tag{4.10}$$

where 'trivial holonomy' means that the group elements g around each triangle have a product equal to the identity 1. Therefore, the Hilbert space $\mathcal{L}^{Q=1}$ has a basis

$$\left\{\left|\alpha_0; g_1, g_2, ...\right\rangle\right\}\Big|_{\text{trivial holonomy}}. \tag{4.11}$$

If the G-action is trivial, then the G-symmetric Hamiltonian can be de-equivariantized as follows. First, φ_3 produces new input data $\{\tilde{w}, \tilde{\delta}, \tilde{T}\}$, where $g, g_1, g_2, g_3 \in G$, by defining

$$\tilde{w}_g = w_{E_{1\,g}}, \tag{4.12}$$

$$\tilde{\delta}_{g_1, g_2, g_3} = \delta_{g_1 g_2 g_3, 1}, \tag{4.13}$$

$$\tilde{T}^{g_1, g_2, (g_1 g_2)^{-1}}_{g_3, (g_1 g_2 g_3)^{-1}, g_2 g_3} = \varphi_3(g_1, g_2, g_3) \big/ w_{g_2 g_3}. \tag{4.14}$$

Then, the Hamiltonian in terms of $\{\tilde{w}, \tilde{\delta}, \tilde{T}\}$ is

$$H = -\sum_v \tilde{Q}_v - \sum_p \tilde{B}_p, \tag{4.15}$$

where $\tilde{B}_p = \frac{1}{n}\sum_g \mathrm{w}_g \tilde{B}_p^g$ for all plaquettes except for p_0, and $\tilde{B}_p^g$ is defined as in equation (3.12) in terms of φ_3, which acts on the degrees of freedom $g_1, g_2, ...$ in the basis (4.11).

At p_0, $\tilde{B}_{p_0} = \frac{1}{n}\sum_g \mathrm{w}_g \tilde{B}_{p_0}^g T_{p_0}^g$, where

$$T_{p_0}^g: \left|\alpha_0; g_1, g_2, ...\right\rangle \mapsto \left|g\alpha_0; g_1, g_2, ...\right\rangle. \tag{4.16}$$

Therefore, the non-local transformation defines a one-to-one correspondence between the G-symmetric LW models and the modified traditional LW models with input data from $\mathcal{V}ec_G^{\varphi_3}$ and $\tilde{B}_{p_0}$ coupled to the local group action $T_{p_0}^g$. The local group action $T_{p_0}^g$ corresponds to a global action in the G-symmetric LW model:

Table 1. Non-local transformation on a disk.

G-symmetric LW model	Traditional LW model coupled to a local action
Global symmetry	A local action on Hamiltonians
Boundary condition	Bulk local quasiparticle
Specified by ρ	With topological charge ρ
Global charge ρ^*	A local charge ρ^* coupled to the quasiparticle

$$T^g\colon \left|\alpha_0,\ \alpha_1,\ldots,\alpha_p,\ldots\right\rangle \mapsto \left|g\alpha_0,\ g\alpha_1,\ldots,g\alpha_p,\ldots\right\rangle. \tag{4.17}$$

Let us apply the non-local transformation on the ground state Φ on the two-sphere. In the transformed traditional LW model, the ground state is the common eigenstate of $\tilde{B}_p^g = 1$, for $p \neq p_0$, and $\tilde{B}_{p_0}^g T_{p_0}^g = 1$, for all $g \in G$. The global constraint in the traditional LW model enforces $\tilde{B}_{p_0}^g = 1$ and hence $T_{p_0}^g = 1$. By the non-local transformation, $T_{p_0}^g = 1$ means that the ground state is invariant under the global symmetry $\{T^g\}$ in the G-symmetric LW model.

Physical theorem[12]: *The G-symmetric LW model with input $\mathcal{M}_n$ realizes a G-SPT with the three-cocycle $\varphi_3 \in H^3(G;\,U(1))$ when φ_3 is G-invariant.*

We did not prove this theorem mathematically because we did not define universality classes of SPT phases mathematically. But physically we summarize the argument above as follows. Each G-invariant three-cocycle φ_3 leads to an SPT because the LW model realizes the trivial TQFT. To understand the local term $T_{p_0}^g$, we map the SPT model to a nontrivial TQFT coupled to a gauge field with a gauge coupling term, where the half-labels represent the gauge field. If we eliminate the gauge coupling term, all half-labels are eliminated as well except the one at the base point. This leaves behind the local term at the base point.

Remark 4.6. The input $6j$-symbols in equations (4.12)–(4.14) are well-defined only when the G-action on φ_3 is trivial. So de-equivariantization works only for trivial G-actions. If the G-action on φ_3 is nontrivial, then the $6j$-symbols are equipped with a G-action, which leads to a LW model with a gauge group action.

4.4. On a disk

Consider further a disk with a smooth boundary, e.g., with the graph in figure 4(a). The non-local transformation leads to the same form of the Hamiltonian as in equation (4.15), but with the second summation over all plaquettes p inside the disk. The degenerate ground states $\Phi(\alpha_\partial)$ in the G-symmetric LW model are parameterized by the half-label α_∂. Now let us reexamine the ground states in the traditional LW model under the non-local transformation.

Take an arbitrary plaquette inside the disk as the origin, denoted by p_0. The ground states are the common eigenstates of $\tilde{B}_p^g = 1$, for $p \neq p_0$ inside the disk, and $\tilde{B}_{p_0}^g T_{p_0}^g = 1$, for all $g \in G$. Due to the presence of the boundary, the global constraint on $\tilde{B}_{p_0}^g$ is released. If $\tilde{B}_{p_0}^g$ transforms under a non-trivial irreducible representation ρ of G, we say there is an elementary quasiparticle (or a topological defect) at p_0 identified by its topological charge ρ. This topological charge is always coupled to a charge which transforms under the dual representation ρ^* of the local group action.

[12] By a physical theorem, we mean that the argument is only rigorous physically. Therefore, physical theorems should be regarded as mathematical conjectures.

The degenerate ground states Φ_ρ are thus parametrized by the charge ρ. Under the non-local transformation, they correspond to the ground states in the G-symmetric LW model, carrying a global charge ρ^* under the global symmetry $\{T^g\}$. Meanwhile, the topological charge ρ of the local quasiparticle in the traditional LW model is mapped to the boundary condition specified by ρ in the G-symmetric LW model. This relation between G-symmetric LW models and LW models coupled to a gauge action is listed in table 1.

For example, take $\mathcal{M}_n$ as the input data, and let $G = \mathbb{Z}_n$. The degenerate ground states can be parameterized by the charge $k = 0, 1, \ldots, n-1$ of $\mathbb{Z}_n$, being the eigenvectors of

$$\tilde{B}^g_{p_0} = \exp\left(\frac{2k\pi g\mathrm{i}}{n}\right),\ T^g_{p_0} = \exp\left(-\frac{2k\pi g\mathrm{i}}{n}\right). \tag{4.18}$$

Such ground states Φ_k are related to $\Phi(\alpha_\partial)$ by the following Fourier transformation

$$\Phi_k = \frac{1}{\sqrt{n}}\sum_{\alpha_\partial}\exp\left(\frac{2k\pi\alpha_\partial\mathrm{i}}{n}\right)\Phi(\alpha_\partial). \tag{4.19}$$

One can verify the identity by applying the action of T^g in equation (4.17) directly.

4.5. On a general closed surface

The de-equivariantization can be applied on an arbitrary closed surface Y in a similar way. The isomorphism in equation (4.10) is replaced by

$$\left\{\alpha_0, \alpha_1, \alpha_2, \ldots\right\}\Big|_{\text{vertex colors}} \cong \left\{\alpha_0; g_1, g_2, \ldots\right\}\Big|_{\text{trivial homotopy and trivial holonomy}}, \tag{4.20}$$

where trivial homotopy means that along any non-contractible loop on the dual-triangulation of the graph, the group elements g multiply to the identity element of G.

G-symmetric LW models are transformed to traditional LW models in the trivial homotopic Hilbert subspace coupled to a local action. The models are well defined because the Hamiltonian is invariant in the trivial homotopic Hilbert subspace.

5. Open questions

We have studied how LW models can be extended to take multi-fusion categories as their input, and how on-site symmetries play a role. There are however still interesting open questions. We mention a few:

(1) Classify $n \times n$ two-matrices.
(2) Prove that the LW model with an indecomposable multi-fusion category input $C = \oplus_{ij} C_{ij}$ realizes the Turaev–Viro TQFT based on C_{ii} for some i.
(3) How to realize symmetry fractionalization, symmetry defects, and gauging with LW models.

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