Fall 2022, MA 571, Final Exam.

Time: 8am, December 12 – 8am, December 13.

Total points: 100. Each problem is worth 10 points.

Send a scan of your solutions to my email (cui177 at purdue dot edu).

1. Let X be a topological space. Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \,|\, x \in X\}$ is closed in $X \times X$.
2. Let $f : X \to Y$ be a continuous function between topological spaces X and Y , and let A be a subset of X. If A is compact, prove that $f(A)$ is compact.
3. Let $f : S^1 \to \mathbb{R}$ be continuous, where S^1 is the unit circle in $\mathbb{R}^2$. Show that there is a point $z \in S^1$ such that $f(z) = f(-z)$.
4. Let X be a compact topological space and Y a Hausdorff topological space. Prove that if $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.
5. Let H^2 be the closed upper hemisphere in the unit sphere S^2, and let $i : H^2 \to S^2$ be the inclusion map. We define an equivalence relation $\sim$ on S^2 by identifying antipodal points:

$$x \sim y \quad \Leftrightarrow \quad x = \pm y, \; x, y \in S^2.$$

   Prove that the induced map $f : H^2/\sim \; \to \; S^2/\sim$ from the map i is a homeomorphism.
6. Endow the set $\mathbb{C}$ of complex numbers with the standard topology on $\mathbb{R}^2$. Show that the map $p : \mathbb{C} \to \mathbb{C} \setminus \{0\}$, defined by $p(z) = e^z$, is a covering map.
7. Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.
8. Let $p : E \to B$ be a covering map, and $p(e_0) = b_0$
   - a). Show that $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.
   - b). Show that the subgroup $p_*(\pi_1(E, e_0))$ in $\pi_1(B, b_0)$ consists of the homotopy classes of loops in B based at b_0 whose liftings in E starting at e_0 are loops.
9. Let $p : E \to B$ be a covering map, where E is compact, path-connected, and locally path-connected. Prove that for each $x \in E$, the set $p^{-1}(p(x))$ is finite, and has cardinality equal to the index of $p_*(\pi_1(E, x))$ in $\pi_1(B, p(x))$.
10. Let X be a metric space. Given any two disjoint closed subsets A, B of X, show that there exist open subsets U, V of X such that $A \subset U,\; B \subset V,\; U \cap V = \emptyset$.