Problem set 4- due October 21. Write clear and complete solutions for each problem. You may send them to the grader Gavin Glenn (email: glenng at purdue dot edu) and cc me (preferred method), or hand it to me in class. Please make your writing clearly legible.

1) Recall that a continuous map between topological spaces $p : X \to Y$ is called a *quotient map* if it is surjective and if a subset V of Y is open if and only if $p^{-1}(V)$ is open in X. Prove that a quotient map $p : X \to Y$ is open (sends any open set in X to an open set in Y) if and only if for every open subset $U \subset X$, the set $p^{-1}(p(U))$ is open in X.

2) Let $\sim$ be an equivalence relation on a space X, $X/\sim$ the set of equivalence classes, and $p : X \to X/\sim$ be the map that sends any point to its equivalence class. Equip $X/\sim$ with the quotient topology induced by p. Prove that if p is open, then the quotient space $X/\sim$ is a Hausdorff space if and only if $\sim$ is a closed subset of the product space $X \times X$.

3) Consider the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, for $n > 0$, with standard topology and let $\mathbb{R}^{n+1} - \{\mathbf{0}\}$ be the subspace obtained by removing the origin. Define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} - \{\mathbf{0}\}$ by setting $\mathbf{x} \sim \mathbf{y}$ if there is a real number $\lambda \in \mathbb{R}$ such that $\mathbf{x} = \lambda \mathbf{y}$. It follows that the set of equivalence classes $(\mathbb{R}^{n+1} - \{\mathbf{0}\})/\sim$ can be identified with the set of lines in $\mathbb{R}^{n+1}$ that go through the origin. Since we have a surjective map

$$p : \mathbb{R}^{n+1} - \{\mathbf{0}\} \to (\mathbb{R}^{n+1} - \{\mathbf{0}\})/\sim$$

that sends a point $\mathbf{x} = (x_1, ..., x_{n+1})$ to its equivalence class $[\mathbf{x}]$ (i.e. to the line going through the origin and $\mathbf{x}$) we can topologize $(\mathbb{R}^{n+1} - \{\mathbf{0}\})/\sim$ with the quotient topology. This space is called the n-th real projective space and is denoted by $\mathbb{R}P^n$.

a) Prove $\mathbb{R}P^1$ is homeomorphic to the circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ equipped with the subspace topology.
b) Let $S^n = \{(x_1, ..., x_n + 1) \in \mathbb{R}^{n+1} : x_1^2 + ... + x_{n+1}^2 = 1\}$ be the n-dimensional sphere equipped with the subspace topology. Let $\sim'$ be the equivalence relation on S^n that identifies antipodal points, namely, $\mathbf{x} \sim' \mathbf{y}$ if $\mathbf{y} = -\mathbf{x}$. Give $S^n/\sim'$ the quotient topology and prove $S^n/\sim'$ is homeomorphic to $\mathbb{R}P^n$.

4) Suppose X is a topological space and A a subspace of X. Recall the interior of A is defined as the uinon of all open sets contained in A. Suppose A is connected. Is the interior of A connected as well? If yes, prove it; if not, give a counterexample and explain it.

Exercise 22.2, 23.2, 23.11