Problem set 7- This assignment will not be collected, but its content will be on the final.

1) Let X be a space and $x_0 \in X$ a point. The underlying set of the fundamental group $\pi_1(X, x_0)$ can be understood as the set of basepoint-preserving homotopy classes of maps of $(S^1, b) \to (X, x_0)$ from the circle S^1 with basepoint $b = (1, 0)$ into (X, x_0). Consider the set $[S^1, X]$ of homotopy classes of maps $S^1 \to X$ with *no restrictions on basepoints*. There is a natural map

$$\psi : \pi_1(X, x_0) \to [S^1, X].$$

Prove that if X is path-connected then

a) ψ is surjective
b) for all loops α and β in X based at x_0, $\psi([\alpha]) = \psi([\beta])$ is and only of $[\alpha]$ and $[\beta]$ are conjugate in $\pi_1(X, x_0)$ (i.e. there exists some $[\gamma] \in \pi_1(X, x_0)$ such that $[\alpha] = [\gamma]^{-1} * [\beta] * [\gamma]$.)

2) Prove that for any space X the following conditions are equivalent:

a) Every map from the circle S^1 into X is homotopic to a constant map
b) Every map from the circle S^1 into X extends to a map from the disk D^2 into X
c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

3) Prove that for any homomorphism of groups $\phi : \pi_1(S^1, b) \to \pi_1(S^1, b)$ you can find a continuous map $f : (S^1, b) \to (S^1, b)$ such that ϕ is induced by f, i.e. $f_* = \phi$.

4) Let X be a space, $A \subset X$ a subspace and $x_0 \in A$.

a) Give an example of some X and A such that the inclusion map $i : A \to X$ induces a map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ which is not injective.
b) Recall that a retraction from a space X into a subspace $A \subset X$ is a map $r : X \to A$ such that the restriction $r|_A : A \to A$ is the identity map. Prove that if there is a retraction $r : X \to A$ then the inclusion map $i : A \to X$ induces an injective map $\pi_1(A, x_0) \to \pi_1(X, x_0)$.
c) Prove there are no retractions $r : X \to A$ when:
i) $X = \mathbb{R}^3$ and A is any subspace homeomorphic to S^1.
ii) $X = S^1 \times D^2$ (the solid torus) and $A = S^1 \times S^1$ (the boundary surface of the solid torus)

5) Suppose E is a path connected space and B is a simply connected space. Prove that any covering map $p : E \to B$ is a homeomorphism.

6) a) Construct a deformation retract from $\mathbb{R}^n - \mathbf{0}$ to $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Compute $\pi_1(\mathbb{R}^n - \mathbf{0})$.

b) Let p_1 and p_2 be *two* distinct points in $\mathbb{R}^3$. Argue why $\mathbb{R}^3 - \{p_1, p_2\}$ is simply connected.

c) Let n be a positive integer and $p_1, ..., p_n$ a sequence of n distinct points on the 2-sphere S^2. Let $X = S^2 - \{p_1, ..., p_n\}$. Calculate $\pi_1(X, x)$ for a choice of basepoint $x \in X$.