1 Differential Forms

A differential form is a linear combination of the base-forms $\omega \in \Omega^k(X)$,

$$\omega = f(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

Where the i_j are chosen in increasing order, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Although not technically the definition, there is no detail lost in defining the value on the standard tangent basis for $\mathbb{R}^n$, namely $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$, to be

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \left(\frac{\partial}{\partial x_{i_1}}, \frac{\partial}{\partial x_{i_2}}, \ldots, \frac{\partial}{\partial x_{i_k}} \right) \equiv +1$$

That is, we get $+1$ if all the indices match. From this definition, all other values can be computed. For intuition we stick to n dimensional forms. Thus we only have one possible form to work with,

$$dx_1 \wedge \cdots \wedge dx_n$$

. If we are given other random vectors $\xi_1, \ldots \xi_n$, we must express them in the $\frac{\partial}{\partial x_i}$ basis, forming the transition matrix

$$M_{\frac{\partial}{\partial x_i} \to \xi_i}$$

Thus to compute

$$dx_1 \wedge \cdots \wedge dx_n \left(\xi_1, \ldots, \xi_n \right)$$

we simply expand each ξ_j in the standard basis, then use the alternating-ness of the form and recognize what pops out is the

$$dx_1 \wedge \cdots \wedge dx_n \left(\xi_1, \ldots, \xi_n \right) = \det \left(M_{\frac{\partial}{\partial x_i} \to \xi_i} \right) dx_1 \wedge \cdots \wedge dx_n \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) = \det \left(M_{\frac{\partial}{\partial x_i} \to \xi_i} \right) (+1)$$

When we deal with manifolds, we may talk about "top-forms" on them, i.e. maximal dimensional forms. So if M is n-dimensional, we have just one dimension of n forms, say it is labeled Ω. Really the only way to work with it is to push it down to $\mathbb{R}^n$, so suppose that we have two charts $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$. Then we know that $\varphi^{-1*}\Omega := \Omega \circ \varphi_*^{-1}$, which will be somewhat demonstrated shortly. Then for the two charts, since Ω must push-down to an n-form, we konw that

$$\varphi_\alpha^{-1*}\Omega = \lambda_\alpha(x) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

$$\varphi_\beta^{-1*}\Omega = \lambda_\beta(x) dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n$$

Where the different letters are signifying the different choice of coordinates. What we are interested in is the comparison of $\lambda_\alpha, \lambda_\beta$. This is done the following way. Form the transition map $\Phi := \varphi_\beta \circ \varphi_\alpha^{-1}$, which is $\mathbb{R}^n \to \mathbb{R}^n$,

$$\Phi(x_1, \ldots, x_n) = (y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n))$$

then use it so push-back $\lambda_\beta(x) dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n$ into "x-coordinates" (really this means into $\varphi_\alpha(U_\alpha)$). That is, consider

$$\Phi^*(\lambda_\beta(x) dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n) = \lambda_\beta(\Phi(x)) dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n(\Phi_* \cdot, \Phi_* \cdots, \ldots, \Phi_* \cdot)$$

Where Φ_* is the map from the "x-coordinate" tangent space (whose basis is $\left\langle \frac{\partial}{\partial x_i} \right\rangle$) to the "y-coordinate" tangent space (whose basis is $\left\langle \frac{\partial}{\partial y_i} \right\rangle$), and we will be shown the "definition" of here now. Given some x-basis vector $\frac{\partial}{\partial x_i}$, we have that

$$\Phi_* \left(\frac{\partial}{\partial x_i} \right) = \frac{\partial y_1}{\partial x_i} \frac{\partial}{\partial y_1} + \frac{\partial y_2}{\partial x_i} \frac{\partial}{\partial x_i} + \cdots + \frac{\partial y_n}{\partial x_n} \frac{\partial}{\partial y_n}$$

In our consideration above, we really need only to check the value of this push back on the standard basis $\frac{\partial}{\partial x_i}$. We compute:

$$\Phi^*(\lambda_\beta(x) dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n) \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) = \lambda_\beta(\Phi(x)) dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n \left(\Phi_* \frac{\partial}{\partial x_1}, \Phi_* \frac{\partial}{\partial x_2}, \ldots, \Phi_* \frac{\partial}{\partial x_n} \right) \tag{1}$$

Here I must resign from detail. Use the definition of $\Phi_*(\frac{\partial}{\partial x_i})$ above and expand each in the equation above. Then use multilinearity of the differential form. Then recognize we can swap entries for (-1) by anti-linearity. Then recognize that

$$\lambda_\beta(\Phi(x)) dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n \left(\Phi_* \frac{\partial}{\partial x_1}, \Phi_* \frac{\partial}{\partial x_2}, \ldots, \Phi_* \frac{\partial}{\partial x_n} \right) = \lambda_\beta() \det \left(Jac(\frac{y_1, \ldots, y_n}{x_1, \ldots, x_n}) \right)$$

This being done though, we remember that

$$\lambda_\alpha(x)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) = \lambda_\alpha()$$

Since these forms are uniquely determined by their values on this input we must have

$$\lambda_\alpha = \lambda_\beta() \det\left(Jac(\frac{y_1, \ldots, y_n}{x_1, \ldots, x_n})\right)$$

We now see why this applies to $\mathbb{R}^n$.

1.1 Application to $\mathbb{R}^n$

Suppose that $U \subset \mathbb{R}^n$ is open. We DEFINE the integral of the differential form

$$\int_U f(x)dx_1 \wedge \cdots \wedge dx_n \equiv \int_U f(x)dx_1 dx_2 \cdots dx_n$$

Now suppose that $V \subset \mathbb{R}^n$ is another open set with coordinates $v_1, \ldots, v_n$. Then we form the transition map

$$F : V \to U, F(v_1, \ldots, v_n) = (x_1(v_1, \ldots, v_n), \ldots, x_n(v_1, \ldots, v_n))$$

Then considering these open sets to be manifolds, we have that

$$\int_U f(x)dx_1 \wedge \cdots \wedge dx_n = \int_V f(F(v))F^*(dx_1 \wedge \cdots \wedge dx_n) = \int_V f(F(v)) \det\left(Jac(\frac{x_1, \ldots, x_n}{v_1, \ldots, v_n})\right) dv_1 \wedge \cdots \wedge dv_n$$

2 Orientation and Riemannian Volume

Now we lay groundwork for volume elements that will be key for our Lie Group Haar Measures.

1. (Orientation Equivalence) Two bases of any vector space $(e_i)_i, (f_j)_j$ are said to be of same orientation if the transition matrix between them has positive determinant, that is, if

   $$\det(M_{e\to f}) > 0$$

   Note that we haven't declared which bases are "positively" oriented, but merely established two equivalence classes of bases. Typically however we will immediately after declare "positive" to be the one of same orientation as the standard $\mathbb{R}^n$ basis, or its push-up to the manifold.

2. (Orientation via charts) Given charts $(U_\alpha, \varphi_\alpha)$, each chart determines an orientation on the tangent spaces on the manifold, just those $T_p(M), p \in U_\alpha$, via declaring that the positively oriented bases at each point are those with same orientation as $\varphi_*^{-1}(\frac{\partial}{\partial x_j}), j = 1, \ldots, n$ (note this is done at every different p, but since our chart is C^∞, these orientations are in some sense "smooth."

   Two charts are said to be "coherently oriented" if the transition map $\varphi_\beta \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ induces an orientation-preserving map of tangent spaces, that is, if $\det(Jac(\varphi_\beta\varphi_\alpha^{-1})) > 0$, since $Jac(\varphi_\beta\varphi_\alpha^{-1})$ is the map that maps

   $$T_x(\mathbb{R}^n_\alpha) \to T_{\varphi_\beta\varphi_\alpha^{-1}(x)}(\mathbb{R}^n_\beta)$$

   The α, β signifying different choices of coordinates.

   Thus, the natural way to define an orientable manifold is to say that it is orientable if it may be covered by coherently-oriented charts.

3. (Orientation via a differential form) An alternative definition of orientation is to say that there is a non-vanishing, top-dimensional differential form Ω on M, that induces a pointwise form Ω_p at every $p \in M$. The reason this determines an orientation is that since given two bases of a tangent space $T_p(M)$, say $[e_{1p}, \ldots, e_{np}], [f_{1p}, \ldots, f_{np}]$, we have, using the same arguments as above, that

   $$\Omega(f_1, \ldots, f_n) = \det(M_{e\to f})\Omega(e_1, \ldots, e_n)$$

   Thus Ω splits the bases of the tangent spaces on whether or not their Ω evaluations have the same sign, giving an orientation. Again we haven't declared which bases are $(+), (-)$, but we will soon.

   This definition is equivalent to the chart definition since if you have the charts above, coherently oriented, each one can induce a differential form on the tangent spaces (pushing up the standard $dx_1 \wedge \cdots \wedge dx_n$), and then use a partition of unity subordinate to the charts to "stitch together" the individual forms.

4. (Riemannian Metric) A Riemannian Metric is nothing more than inner product on all the tangent spaces of the points of the manifold, i.e. Φ assigns to each point p an inner-product Φ_p an inner-product on $T_p(M)$. Since given a neighborhood of p and a chart (U, φ) this chart induces a standard tangent-space basis $E_{1p}, \ldots, E_{np}$ on the nearby tangent spaces, we have at each point $u \in U$, that Φ_u is determined by

$$\Phi_u(E_{iu}, E_{ju}) \equiv g_{ij}(u)$$

The metric is said to be C^∞ if for fixed i, j the function g_{ij} is a smooth function of u (of course only on the neighborhood of p).

Theorem 1. *If M is smooth manifold together with a Riemannian metric and an orientation (i.e. a choice of "(+)" bases, then using the orientation given we may create a differential form Ω such that $\Omega(F_{1p}, \ldots, F_{np}) = +1$ on every positively oriented orthonormal basis of the respective tangent spaces $T_p(M), p \in M$. As a consequence of this declaration, we discover that the value on the chart basis $E_{ip}, i = 1, \ldots, n$ above is given*

$$\Omega(E_{1p}, \ldots, E_{np}) = \sqrt{\det(g_{ij}(p))}$$

where $g_{ij}(p)$ is referring to the full matrix of entries for the inner product Φ_p.

Corallary 2. *If we have an oriented Riemannian manifold, then there exists the Ω above. This Ω has push-down to $\mathbb{R}^n$ given by*

$$\varphi^{-1*}\Omega = \sqrt{\det(g_{ij}(p))}dx_1 \wedge \cdots \wedge dx_n$$