Complex numbers basics

Solving the equation $x^2 + 1 = 0$, we see that $x = \pm\sqrt{-1}$. Since $\sqrt{-1}$ isn't well-defined as a "real" number, we define the "imaginary number"

$$i = \sqrt{-1},$$

so that the roots of $x^2 + 1 = 0$ are $x = \pm i$. With this, we can actually *factor*

$$x^2 + 1 = (x - i)(x + i).$$

Remark. Some engineering and physics texts use $j = \sqrt{-1}$ instead of i.

Definition. Complex numbers, usually denoted with z or w, are numbers of the form

$$z = a + bi \quad \text{(and/or } z = x + yi\text{)}, \quad \text{where} \quad a, b, \in \mathbb{R} \quad \text{(or } x, y \in \mathbb{R}\text{)}.$$

If $z = a + bi$, we say that a and b are the real part (of z) and the imaginary part of z, respectively. Symbollicaly these are written with

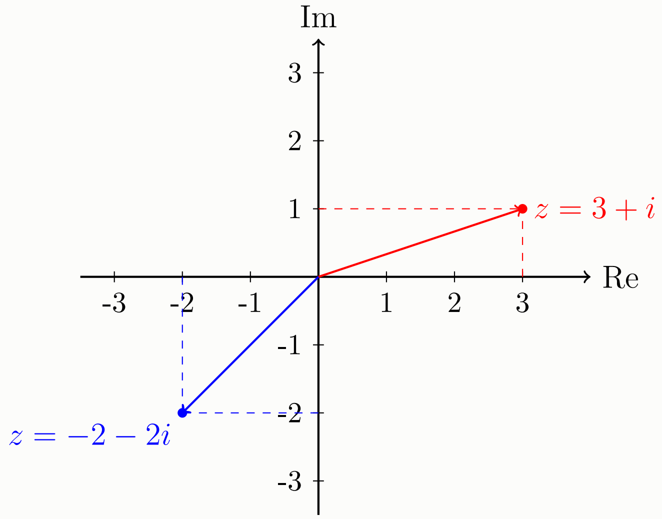
$$z = a + bi \quad \Longrightarrow \quad \mathrm{Re}(z) = a \quad \text{and} \quad \mathrm{Im}(z) = b.$$

Frequently you may see "gothic" letters used instead, i.e., writing

$$\Re(z) = \mathrm{Re}(z) \qquad \text{and} \qquad \Im(z) = \mathrm{Im}(z).$$

Naturally, the complex number $0 + 0i$ is usually just written as "0".

We associate real numbers to points on a line, and complex numbers to points on a plane, called the Complex Plane:



Operations with complex numbers:

(1) (Equality)

$$a_1 + b_1 i = a_2 + b_2 i \quad \Longleftrightarrow \quad a_1 = a_2 \text{ and } b_1 = b_2.$$

$$\text{Ex:} \qquad 2 + 3i = 2 + 3i, \qquad 1 + 2i \neq 1 + 5i.$$

(2) (Addition) Done "componentwise":

$$a_1 + b_1 i + a_2 + b_2 i = (a_1 + a_2) + (b_1 + b_2)i.$$

$$(3 + 7i) + (2 + 5i) = 5 + 12i,$$

$$(1 - \sqrt{3}i) + (2 + \sqrt{2}i) = 3 + (\sqrt{2} - \sqrt{3})i.$$

(3) (Multiplication) This is essentially just "foil-ing" using $i^2 = -1$:

$$\begin{aligned}(a_1 + b_1 i)(a_2 + b_2 i) &= a_1 a_2 + (a_1 b_2 + a_2 b_1)i + b_1 b_2 i^2 \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i.\end{aligned}$$

$$(2 + 3i)(1 + i) = 2 + 2i + 3i + 3i^2 = -1 + 5i,$$

$$(1 + \sqrt{3}\,i)(1 - \sqrt{3}\,i) = 1 - (\sqrt{3})^2 i^2 = 4.$$

(4) (Absolute value) Sometimes called the "norm"; this is the same as the vector length from $(0, 0)$ to (a, b):

$$z = a + bi \quad \Longrightarrow \quad |z| \overset{\text{DEF}}{=} \sqrt{a^2 + b^2}$$

(5) (**Conjugation**) If $z = a + bi$, then $\bar{z} \overset{\text{DEF}}{=} a - bi$ is the conjugate of z.
   (a) When speaking we might say "z-bar" for $\bar{z}$.
   (b) In essence, conjugation flips $i \to -i$, which flips numbers across the x-axis.

$$z = -2 + 3i \quad \Longrightarrow \quad \bar{z} = -2 - 3i, \qquad z = 3 + i \quad \Longrightarrow \quad \bar{z} = 3 - i.$$

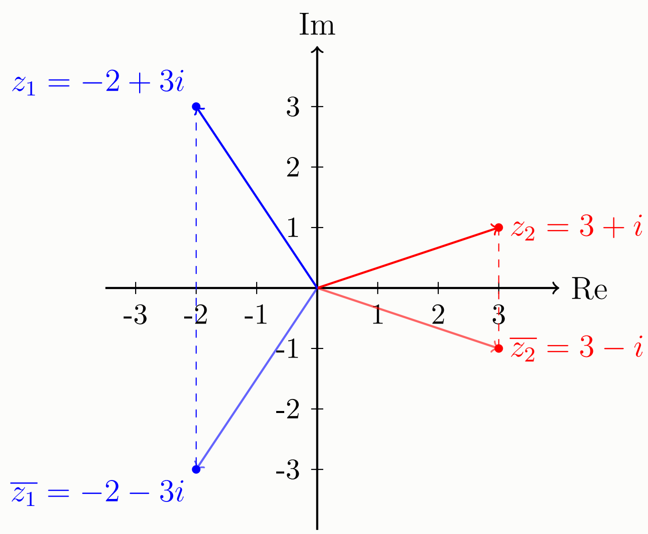


FIGURE 1. Conjugates

(6) (Division, by example) Generally we don't like to have imaginary numbers in denominators, so we can convert something like $z = 1/(3 + 4i)$ back into a usual complex number. If α, β make

$$\frac{1}{3 + 4i} = \alpha + \beta i, \qquad \text{then} \qquad (3 + 4i)(\alpha + \beta i) = (3 + 4i)\frac{1}{3 + 4i} = 1 + 0i.$$

Then, expanding $(3+4i)(\alpha+\beta i)$, we have

$$(3+4i)(\alpha+\beta i) = 1+0i \quad \Longrightarrow \quad (3\alpha-4\beta)+(3\beta+4\alpha)i = 1+0i$$

$$\Longrightarrow \begin{cases} 3\alpha-4\beta=1 \\ 4\alpha+3\beta=0 \end{cases} \quad \Longrightarrow \quad \cdots \quad \Longrightarrow \begin{cases} \alpha=\frac{3}{25} \\ \beta=-\frac{4}{25} \end{cases}.$$

Thus, in total we have

$$\frac{1}{3+4i} = \frac{3}{25} - \frac{4}{25}i.$$

It would be tedious to do this each time; thankfully there is a general formula:

$$\frac{1}{a+bi} = \left(\frac{a}{a^2+b^2}\right) + \left(\frac{-b}{a^2+b^2}\right)i = \frac{a-bi}{a^2+b^2}.$$

(7) **(A very useful formula):** Looking at the formula for $\frac{1}{a+bi}$ just above, we see

$$z = a+bi \quad \Longrightarrow \quad \frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{\bar{z}}{|z|^2}.$$

This fact that $1/z = \bar{z}/|z|^2$ is a super useful formula for complex numbers, and is used frequently in many contexts/applications.

We could verify it another way "directly":

$$\begin{aligned} z = a+bi \quad \Longrightarrow \quad z\bar{z} = (a+bi)(a-bi) &= a^2 + (ab-ab)i - b^2 i^2 \\ &= a^2+b^2 \\ &= |z|^2. \end{aligned}$$

Thus

$$z\bar{z} = |z|^2, \qquad \text{which means that} \qquad \frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$