Linear dependence

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors either all in $\mathbb{R}^2$ or all in $\mathbb{R}^3$. The following statements are all equivalent expressions of the idea

"At least one of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is a linear combination of the others."

(1) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent.
(2) The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a "linearly dependent set".
(3) At least one of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is in the span of the other two. (Ex: $\mathbf{u} \in \text{span}\{\mathbf{v}, \mathbf{w}\}$ or $\mathbf{v} \in \text{span}\{\mathbf{u}, \mathbf{w}\}$.)
(4) There is a tuple (a, b, c) of scalars, with $(a, b, c) \neq (0, 0, 0)$, such that the linear combination

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}.$$

(5) The vector equation/system of linear equations

$$\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \text{or, same as:} \qquad \mathbf{A}\mathbf{x} = \mathbf{0},$$

has (in addition to the "trivial solution" $\mathbf{x} = (0, 0, 0)$), some other solution $\mathbf{x} = (x_1, x_2, x_3)$ such that $\mathbf{x} \neq \mathbf{0}$.

Linear independence

The following statements are all equivalent to the "negation" of the above, namely:

"No one of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is a linear combination of the others."

(1) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly indpendent.
(2) The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a "linearly independent set".
(3) No one of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is in the span of the other two.
(4) The *only* tuple (a, b, c) of scalars which makes the linear combination

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$$

is the tuple $(a, b, c) = (0, 0, 0)$.
(5) The vector equation/system of linear equations

$$\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{or, same as:} \qquad \mathbf{A}\mathbf{x} = \mathbf{0},$$

has *sole* solution $\mathbf{x} = \mathbf{0} = (0, 0, 0)$ (a.k.a. "the trivial solution").