These few pages collect much of the vocabulary we have seen around linear dependence and independence.

Linear dependence (3 vectors)

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors all of the same size, and let

$$\mathbf{A} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}.$$

Definition: $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent if *at least one* of them is a linear combination of the others.

The following statements are all equivalent expressions of the idea/definition of linear dependence:

(1) The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a "linearly dependent set".

(2) At least one of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is in the span of the other two. (ex: $\mathbf{u} \in \operatorname{span}\{\mathbf{v}, \mathbf{w}\}$, or $\mathbf{v} \in \operatorname{span}\{\mathbf{u}, \mathbf{w}\}$, etc..)

(3) There are scalars $(a, b, c) \neq (0, 0, 0)$, such that the linear combination

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}.$$

(4) The vector equation $\mathbf{A}\mathbf{x} = \mathbf{0}$,

$$a.k.a. \qquad \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

has *some* solution $\mathbf{x} = (x_1, x_2, x_3)$ *other than* the trivial solution $\mathbf{x} = (0, 0, 0) = \mathbf{0}$.

(5) The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is not a *basis* for $\operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. [Note: By definition, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is* a spanning set for $\operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$; they don't make a basis because they are lin. depen.]

(6) The dimension $\dim(\operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}) < 3$.

(7) Because $\mathbf{u}, \mathbf{v}, \mathbf{w}$ make up the columns of $\mathbf{A}$, one has $\dim(\operatorname{col}(\mathbf{A})) < 3$. This is the same as saying $\operatorname{rank}(\mathbf{A}) < 3$.

(8) $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is *not* a basis for the set $\operatorname{col}(\mathbf{A})$.

(9*) The determinant $\det\left(\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}\right) = 0$.

[Linear independence on next page]

*Only works if $\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$ is a square matrix (so, in this case a 3×3).

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$$\mathbf{A} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}.$$

Definition: $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent if *no one* of them is a linear combination of the others.

The following statements are all equivalent to the idea of linear independence, and are all essentially "negations" of the above "dependence statements":

(1) The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a "linearly independent set".

(2) No one of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is in the span of the other two.

(3) The *only* tuple of scalars (a, b, c) which makes the linear combination

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$$

is the tuple $(a, b, c) = (0, 0, 0)$.

(4) The *only* solution to the vector equation $\mathbf{Ax} = \mathbf{0}$,

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(7) Because $\mathbf{u}, \mathbf{v}, \mathbf{w}$ make up the columns of $\mathbf{A}$, one has $\dim(\operatorname{col}(\mathbf{A})) = 3$. This is the same as saying $\operatorname{rank}(\mathbf{A}) = 3$.

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[Linear dependence for more vectors on next page]

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Linear dependence (any number of vectors)

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors all of the same size (say $n \times 1$, so they are in $\mathbb{R}^n$), and let

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix}.$$

Definition: $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent if *at least one* of them is a linear combination of the others.

(1) The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a "linearly dependent set".

(2) At least one of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is in the span of the others. (ex: $\mathbf{v}_2 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_5\}$, or $\mathbf{v}_1 \in \operatorname{span}\{\mathbf{v}_2, \ldots, \mathbf{v}_k\}$, etc..)

(3) There is a tuple of scalars $(x_1, \ldots, x_k) \neq (0, \ldots, 0)$ (k-many 0s), such that the linear combination

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}.$$

(4) The vector equation $\mathbf{A}\mathbf{x} = \mathbf{0}$,

a.k.a. $$\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (\mathbf{A}_{n\times k} \text{ and } \mathbf{x}_{k\times 1} \text{ means } \mathbf{0} \text{ is } n \times 1)$$

has *some* solution $\mathbf{x} = (x_1, \ldots, x_k)$ *other than* the trivial solution $\mathbf{x} = (0, \ldots, 0) = \mathbf{0}$.

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(6) The dimension $\dim(\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}) < k$.

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[Linear independence on next page]

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