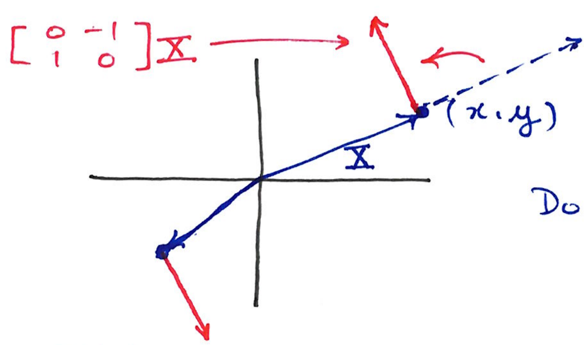
- $\underline{X}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ gives a position vector/point (x,y) over time
- $\underline{X}'(t) = \begin{bmatrix} x' \\ y' \end{bmatrix}$ is the "velocity vector" at point (x,y)

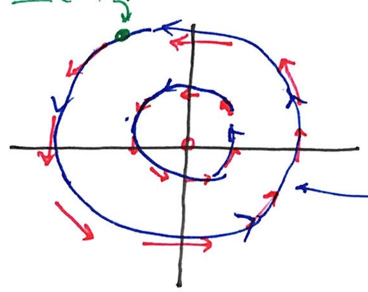
  (draw base of $\underline{X}'$ at $\underline{X}$)
- homogeneous vector ODE $\underline{X}' = \underline{P}\,\underline{X}$ tells you how to make or draw/plot velocity vectors at each point: by rotating/scaling/tranforming $\underline{X} \longrightarrow \underline{P}\underline{X}$.

$\underline{Ex}$: $\underline{X}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\underline{X}$ represents $90°$ turn.



Do this at every point ...

initial value $\underline{X}(0)$



"velocity vector field"
(for ODE $\underline{X}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\underline{X}$)

solution curve $\underline{X}(t)$

a solution $\underline{X}(t)$ for $\underline{X}' = \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{P}\underline{X}$ is a (parametric) curve that has velocity vectors "according to the vector field $\underline{X}' = \underset{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}{\underline{P}}\underline{X}$" (specific curve depends on where you start ... initial conditions/values)

- $\underline{\text{Special thing}}$: if $\underline{X}'$ "points in the same direction" as position $\underline{X}$, we get a kind of "feedback" causing $\underline{X}(t)$ curve to $\underline{\text{stay}}$ along same line. $\underline{X}' = \underline{k\underline{X}}$, but also $\underline{X}' = \underline{P}\,\underline{X}$ ... $\Rightarrow \underbrace{\underline{P}\underline{X} = k\underline{X}}_{\text{eigenvector!}}$

Full Example $\underline{X}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \underline{X} \quad \left(\underline{X} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \right)$

Sketch solution curves for this vector ODE

Part I : Find our "straight line solutions"

Step 1 Compute eigenvalues $\det(\underline{A} - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix}$

$= (3-\lambda)(-3-\lambda) + 5 = \lambda^2 - 4 = 0$

$\Rightarrow$ eigenvals are $\lambda = -2$ (multi 1)

$\lambda = 2$ (" " 1)

Step 2 Find eigenvectors

$\underline{\lambda_1 = -2}$ Solve $(\underline{A} - (-2)I)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 5 & -1 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad 5a - b = 0$

$a = \frac{1}{5} b$

eigenvectors are $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{5} b \\ b \end{bmatrix} = b\begin{bmatrix} 1/5 \\ 1 \end{bmatrix}$. Pick $b = 5$,

set $\underline{v_1} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

$\underline{\lambda_2 = 2}$ ... eigenvector $\underline{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Step 3 "Straight line solutions"

$\underline{X}_1 = e^{\lambda_1 t}\underline{v_1} = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad \left(= \begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix} \right)$

$\underline{X}_2 = e^{\lambda_2 t}\underline{v_2} = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{X}_1' = (e^{\lambda_1 t}\underline{v_1})' = (e^{\lambda_1 t})'\overbrace{\begin{bmatrix} 1 \\ 5 \end{bmatrix}}^{v_1} = \lambda_1 e^{\lambda_1 t}\underline{v_1} = \underline{(-2)e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}}$

$\begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\underline{X}_1 = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}(\underbrace{e^{\lambda_1 t}}_{\text{scalar}}\underline{v_1}) = e^{-2t}\begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = e^{-2t}\begin{bmatrix} -2 \\ -10 \end{bmatrix}$

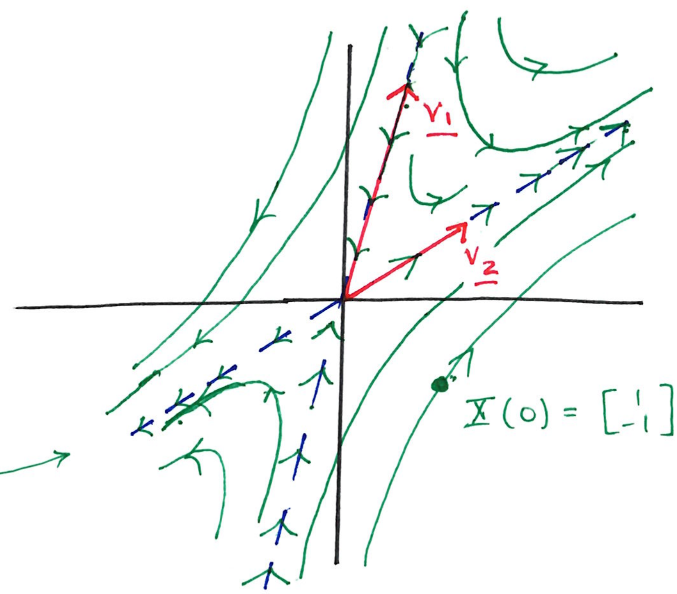
$= \underline{(-2)e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}}$

so indeed $\underline{X}_1 = e^{\lambda_1 t}\underline{v_1}$ is a solution for $\underline{X}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\underline{X}$.

Part II. Drawing some solutions for $\underline{X}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \underline{X}$

$\lambda_1 = -2, \quad \underline{v_1} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\lambda_2 = 2, \quad \underline{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



(not to scale)

like an asymptote

1) Plot eigenvectors & their spans

★: pos./neg. eigenvalue indicates "away/toward" the origin along that vector/span.

"Straight line solutions" $\underline{X}_1 = e^{\lambda_1 t} \underline{v_1} = e^{-2t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\underline{X}_2 = e^{\lambda_2 t} \underline{v_2} = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2) "~~Interopal~~ Interpolation" between straight line solutions to draw other solutions.

General solution is $\underline{X} = c_1 \underline{X}_1 + c_2 \underline{X}_2$.

("vector form") → $= c_1 e^{-2t} \underline{\begin{bmatrix} 1 \\ 5 \end{bmatrix}} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} c_1 e^{-2t} + c_2 e^{2t} \\ 5c_1 e^{-2t} + c_2 e^{2t} \end{bmatrix}$

CAUTION: Because c_1 appears in multiple rows, do not change, because $5c_1 \to c_1$ would make solution

$c_1 e^{-2t} \underline{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

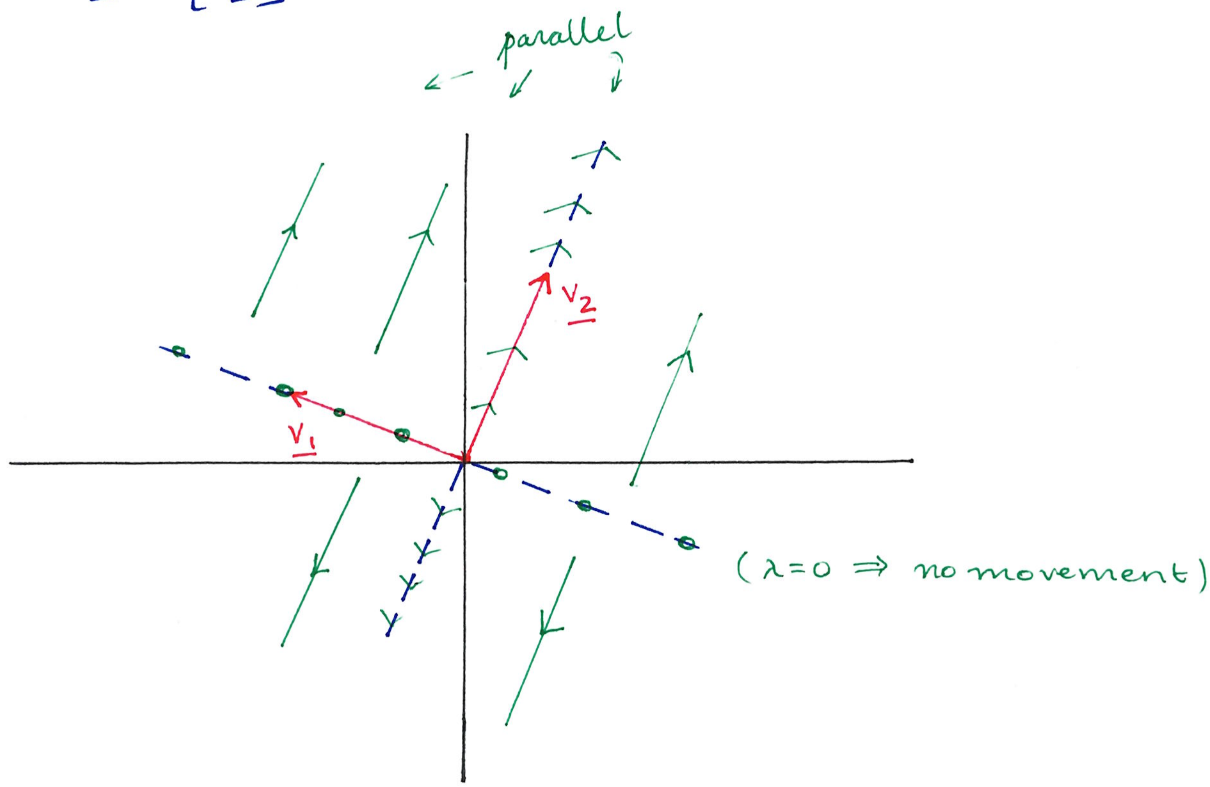
different direction

$\mathcal{E}x$: $X' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X$.

We already found e-values/vectors:

$\lambda_1 = 0$, $\underline{v_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda_2 = 5$, $\underline{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



1) Plot eigenvectors + spans.

2) Straight line solutions are

$$X_1 = e^{\lambda_1 t}\underline{v_1} = e^{0t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$X_2 = e^{\lambda_2 t}\underline{v_2} = e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

3) "Interpolate": general solution is

$$X = c_1 X_1 + c_2 X_2$$
$$= c_1\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} -2c_1 + c_2 e^{5t} \\ c_1 + 2c_2 e^{5t} \end{bmatrix}$$

☆ Presently we only know how to draw a couple examples. Will revisit drawing after computational methods.