Clarifying the Wronskian

Recall that

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

By its definition, if y_1, y_2, y_3 are functions of x, then $W(y_1, y_2, y_3)$ is also a function of x

We saw this in an example with 2 funcs.:

$$W(e^x, e^{3x}) = \begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix} = 3e^{4x} - e^{4x}$$
$$= \underline{2e^{4x}},$$

which is certainly a function of x.

Thus, sometimes we might write

$$W(e^x, e^{3x})(x)$$

to emphasize this dependence on x.

Similarly, with 3 functions we can write either $W(y_1, y_2, y_3)$ <u>or</u> $W(y_1, y_2, y_3)(x)$

The following is a good test-style question:

If $f_1 = e^{3x}$, $f_2 = xe^{3x}$, $f_3 = x^2e^{3x}$, compute $W(f_1, f_2, f_3)(0)$.

For this, we want to evaluate $W(e^{3x}, xe^{3x}, x^2e^{3x})$ at the input $x=0$, so we first "setup" the Wronskian.

$$(x^2e^{3x})' = 2xe^{3x} + 3x^2e^{3x} = \underline{(2x+3x^2)e^{3x}}$$

$$(x^2e^{3x})'' = (2+6x)e^{3x} + (6x+9x^2)e^{3x}$$
$$= \underline{(2+12x+9x^2)e^{3x}}$$

$$W(e^{3x}, xe^{3x}, x^2e^{3x})$$
$$= \begin{vmatrix} e^{3x} & xe^{3x} & x^2e^{3x} \\ 3e^{3x} & (1+3x)e^{3x} & (2x+3x^2)e^{3x} \\ 9e^{3x} & (6+9x)e^{3x} & (2+12x+9x^2)e^{3x} \end{vmatrix}$$

Now, it would be quite tedious to expand this entire determinant, but in fact we don't exactly:

Because we only want the determinant at $x=0$, we plug in $x=0$, then compute the determinant.

So
$$\begin{vmatrix} e^{3x} & xe^{3x} & x^2e^{3x} \\ 3e^{3x} & (1+3x)e^{3x} & (2x+3x^2)e^{3x} \\ 9e^{3x} & (6+9x)e^{3x} & (2+12x+9x^2)e^{3x} \end{vmatrix}$$

becomes

(expand along row 1)

$$\begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 6 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 \\ 6 & 2 \end{vmatrix} + 0 \cdots$$

$$= (2-0) = \boxed{2}$$

In total,

$$\boxed{W(e^{3x}, xe^{3x}, x^2e^{3x})(0) = 2}$$

The Wronskian and linear independence

The following theorem contains the core idea/fact about the Wronskian

Thm: Let I be some interval (e.g. $(0,\infty)$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(-\infty, \infty)$, etc.) and let y_1, y_2, y_3 be functions of x defined on said interval I.

The Wronskian $W(y_1, y_2, y_3)$, a.k.a.

$$W(y_1, y_2, y_3)(x) \overset{DEF}{=} \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$ is a

function of x, also defined on I.

Case 1: If this resulting function $W(y_1, y_2, y_3)$ is the "$\emptyset$-function on I"*, then y_1, y_2, y_3 are linearly dependent.

Case 2: If $W(y_1, y_2, y_3)$ (aka. $W(y_1, y_2, y_3)(x)$) is not the constant $\emptyset$ function on I**, then y_1, y_2, y_3 are linearly independent

* that is, the function with domain I and constant range $\emptyset$.

** to not be the constant $\emptyset$ function on I, all you need is some x-value $x_0 \in I$ for which $W(y_1, y_2, y_3)(x_0) \neq 0$

Naturally, the theorem holds similarly if you have just two funcs. y_1, y_2, or four y_1, y_2, y_3, y_4, or five, etc. ... provided that you use the proper size/shape Wronskian. Ex: for four y_1, y_2, y_3, y_4, you use

$$\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}, \quad \text{and so on...}$$