Sec 4.3 Bases & Null space.

Def a set of vectors $\{\underline{v_1}, \dots, \underline{v_k}\}$ is a basis for a set V
if both : 1) $\text{span}\{\underline{v_1}, \dots, \underline{v_k}\} = V$
2) $\{\underline{v_1}, \dots, \underline{v_k}\}$ is a lin. indep set.

Note: adding/Removing vectors from a set can change depen/indep — need to check again.

Def : The standard ordered basis for $\mathbb{R}^n$ is

$$\{\underline{e_1}, \underline{e_2}, \dots, \underline{e_n}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

in $\mathbb{R}^3$ $\{\underline{e_1}, \underline{e_2}, \underline{e_3}\} = \{\hat{i}, \hat{j}, \hat{k}\}$

Sets can have many different bases... how do we find them?

Def : If $\underline{A}$ $(n \times k)$, then null space

$$\text{null}(\underline{A}) = \{ \underline{x} \in \underset{(k\times 1)}{\mathbb{R}^k} : \underline{A}\underline{x} = \underline{0}_{n\times 1} \}$$

$(n\times k)(k\times 1) \longrightarrow (n \times 1)$

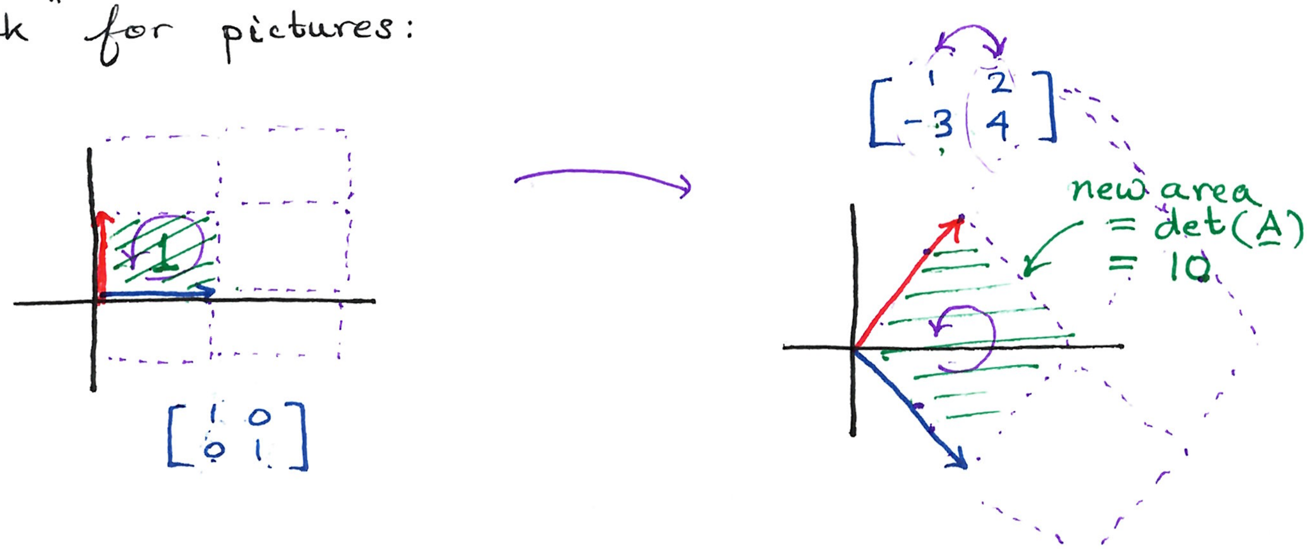
Ex : Compute a basis for null($\underline{A}$), where

$$\underline{A} = \begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix}_{(3\times 4)}$$

Looking for solutions to system $\underline{A}\underline{x} = \underline{0}$

$$\underset{3\times 4}{\begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

"Break" for pictures:



Going to REF in example:

$$\begin{array}{c} \left[\begin{array}{cccc|c} \boxed{1} & 0 & -3 & -2 & 0 \\ 0 & \boxed{1} & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ x_1 \quad x_2 \quad \underbrace{x_3 \quad x_4}_{} \text{ free} \end{array}$$

$x_3 = s$ free params

$x_4 = t$

"$s, t \in \mathbb{R}$"

solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s+2t \\ 4s-3t \\ s \\ t \end{bmatrix} = s \underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{\underline{u}} + t \underbrace{\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\underline{v}} \quad s, t \in \mathbb{R}$$

$$\text{null}(\underline{A}) = \left\{ s \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$\text{null}(\underline{A}) = \text{span}\left\{ \underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{\underline{u}}, \underbrace{\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\underline{v}} \right\}$$

- Is null($\underline{A}$) a subspace of $\mathbb{R}^4$? Yes
- Are $\underline{u}, \underline{v}$ lin indep? Yes.
- So, null($\underline{A}$) = span$\{\underline{u}, \underline{v}\}$ & $\{\underline{u}, \underline{v}\}$ lin indep.

  so $\{\underline{u}, \underline{v}\}$ is a basis for null($\underline{A}$)