Dealing with generalized eigenvectors.

Definition. A vector $\mathbf{v}$ is a "genuine" eigenvector for $\mathbf{A}$, with eigenvalue λ, if

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}.$$

If we instead have

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}\neq\mathbf{0} \qquad \text{but} \qquad (\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}=\mathbf{0},$$

then $\mathbf{v}$ is a (2-step) generalized eigenvector. If

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}\neq\mathbf{0}, \qquad (\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}\neq\mathbf{0}, \qquad \text{but} \qquad (\mathbf{A}-\lambda\mathbf{I})^3\mathbf{v}=\mathbf{0},$$

then $\mathbf{v}$ is a (3-step) generalized eigenvector, and so forth.

Definition (multiplicity/defect). Let $\mathbf{A}$ be a matrix and let λ be an eigenvalue of $\mathbf{A}$. Thus, λ is a root of the characteristic polynomial $|\mathbf{A}-\lambda\mathbf{I}|$.

(1) The algebraic multiplicity of λ is the number of times it occurs as a root of $|\mathbf{A}-\lambda\mathbf{I}|$.

(2) The geometric multiplicity of λ is $\dim(E_\lambda)=\dim(\text{null}(\mathbf{A}-\lambda\mathbf{I}))$.

(3) The defect of λ is $d=(\text{alg. mult. of }\lambda)-(\text{geom. mult. of }\lambda)$.

(4) If λ has defect 0, we can say that λ "is complete", or that "λ has a complete set of eigenvectors".

Note: Sometimes we can use the terminology "defective" or "nondefective" as adjectives for the matrix $\mathbf{A}$ itself: That is, if $\mathbf{A}$ has some λ with defect $d>0$, we might say "$\mathbf{A}$ is a defective matrix."

Solutions with generalized eigenvectors (defect = 1).

In solving $\mathbf{X}'=\mathbf{AX}$, suppose that the eigenvalue λ occurs twice (has alg. mult. 2) and has defect $d=1$. Then we infer that (geom. mult. of λ) is (defect d) $-$ (alg. mult.) $=1$. Thus

$$\dim E_\lambda = 1, \qquad \text{and we pick out genuine eigenvector} \quad \mathbf{v}_1.$$

Then our two basis solutions are

$$\begin{aligned} \mathbf{X}_1 &= e^{\lambda t}\mathbf{v}_1, & (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1 &= \mathbf{0}, \\ \mathbf{X}_2 &= e^{\lambda t}(t\mathbf{v}_1+\mathbf{v}_2), & (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 &= \mathbf{v}_1. \end{aligned}$$

Solutions with generalized eigenvectors (defect $\geq$ 2).

If λ occurs 3 times, and we have defect $d=2$, with $\mathbf{v}_1$ a genuine eig.vect., then we need to make a "3-step chain" of eig.vects.. Namely, we need $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ with

$$\begin{cases} (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1=\mathbf{0} & (\mathbf{v}_1 \text{ is genuine eig.vect.}), \\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2=\mathbf{v}_1 & (\mathbf{v}_2 \text{ is 2-step eig.vect.}), \\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3=\mathbf{v}_2 & (\mathbf{v}_3 \text{ is 3-step eig.vect.}). \end{cases}$$

There are two methods for making this "3-chain".

Method 1:

1. Solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ to pick $\mathbf{v}_1\neq\mathbf{0}$ (we assume this is already done).
2. Solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{v}_1$ to find/pick $\mathbf{v}_2$.
3. Solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{v}_2$ to find/pick $\mathbf{v}_3$.

Method 2 (Preferred/Easier):

1. Find a $\mathbf{v}\neq\mathbf{0}$ such that $(\mathbf{A}-\lambda\mathbf{I})^3\mathbf{v}=\mathbf{0}$, and set that as "$\mathbf{v}_3$".
2. Check that both $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3\neq\mathbf{0}$ and $(\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}_3\neq\mathbf{0}$. If one of those is $\mathbf{0}$, restart with a different $\mathbf{v}_3$.
3. If neither is $\mathbf{0}$, then (re)set

$$\mathbf{v}_2 \overset{\text{DEF}}{=} (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3, \qquad \mathbf{v}_1 \overset{\text{DEF}}{=} (\mathbf{A}-\lambda\mathbf{I})^2\mathbf{v}_3 = (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2.$$

   Then these $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ make the desired "3-chain".

Once we have the desired "3-chain" with $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, our basis solutions are

$$\begin{aligned}
\mathbf{X}_1 &= e^{\lambda t}\mathbf{v}_1, & (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1 &= \mathbf{0},\\
\mathbf{X}_2 &= e^{\lambda t}\big(t\mathbf{v}_1+\mathbf{v}_2\big), & (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 &= \mathbf{v}_1,\\
\mathbf{X}_3 &= e^{\lambda t}\big(\tfrac{1}{2}t^2\mathbf{v}_1+t\mathbf{v}_2+\mathbf{v}_3\big), & (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3 &= \mathbf{v}_2.
\end{aligned}$$

Remark. Using Method 2 above, it is easy to see how we could extend the method to handle making a "4-chain", "5-chain", etc.. For instance, with a 4-chain we would add $\mathbf{v}_4$ and another basis solution

$$\mathbf{X}_4 = e^{\lambda t}\big(\tfrac{1}{6}t^3\mathbf{v}_1+\tfrac{1}{2}t^2\mathbf{v}_2+t\mathbf{v}_3+\mathbf{v}_4\big), \qquad (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_4 = \mathbf{v}_3.$$

Ex : $\underline{A} = \begin{bmatrix} -11 & 0 & -4 \\ -1 & -9 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, $|\underline{A} - \lambda I| = 0 \Rightarrow \ldots$

$\ldots \Rightarrow \lambda = -9, -9, -9$

(algebraic multiplicity 3)

Solve $\left(\underline{A} - (-9)I\right)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

augmented matrix

$$\left[\begin{array}{ccc|c} -2 & 0 & -4 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

$$R_1 \xleftrightarrow{\text{swap}} R_2 \quad \left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ -2 & 0 & -4 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} R_1 \to (-1)R_1 \\ R_2 \to R_2 + 2R_3 \end{array}\right\} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

$$R_2 \xleftrightarrow{\text{swap}} R_3 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \to R_2 - R_1 \quad \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

b free, a, c leading $\left\{ \begin{array}{l} a + c = 0 \Rightarrow a = 0 \\ c = 0 \end{array} \right.$

so solutions for $\left(\underline{A} - (-9)I\right)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$ are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

only one basis vector, pick $b = 1$, $\underline{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Because $\lambda = -9$ has alg. mult. 3, but we found one lin. indep eigenvector (the $\underline{v}_1$ we just found),

$\dim(\mathcal{E}_{-9}) = 1 \Rightarrow$ geom mult. 1, and so $\lambda = -9$ has defect $d = 3-1 = 2$. So we need to find 2 generalized e-vects $\underline{v}_2, \underline{v}_3$

Use (method 2) to make a "chain" of length 3, namely just want $\underline{v}_1, \underline{v}_2, \underline{v}_3 \neq 0$ such that

$$\begin{cases} (\underline{A} - \lambda I)^3 \underline{v}_3 = 0 \\ \underline{v}_2 := (\underline{A} - \lambda I)\underline{v}_3 & (\text{needs to be } \neq 0) \\ \underline{v}_1 := (\underline{A} - \lambda I)\underline{v}_2 & (\quad '' \quad '' \quad '' \quad '' \quad) \end{cases}$$

Using $\underline{A}$ from example and $\lambda = -9$, we find that

$$(\underline{A} - (-9)I)^3 \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

so a, b, c free. Pick $a = 1$, $b = 0$, $c = 0$, $\underline{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then

$$\underline{v}_2 = (\underline{A} - (-9)I)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -1 \\ +1 \end{bmatrix} \quad (\neq 0)$$

$$\underline{v}_1 = (\underline{A} - (-9)I)\begin{bmatrix} -2 \\ -1 \\ +1 \end{bmatrix}$$

$$= \cdots = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(which happens to be the same $\underline{v}_1$ we found earlier, but it isn't always like this)

What if we picked $a=0$, $b=1$, $c=0$ and made $\underline{v}_3 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$?

Then
$$\begin{aligned}\underline{v}_2 &= (\underline{A} - (-9)I)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ +1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\end{aligned}$$

but we said $\underline{v}_2$ can't be 0. ($\underline{v}_1$ would also be $=0$ here, which also can't happen)

Hence we shouldn't use the choice $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

* for this problem.

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ might work in other problems.

So we have $\underline{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Solutions
$$\mathbf{X}_1(t) := e^{\lambda t}\underline{v}_1 = e^{-9t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{aligned}\mathbf{X}_1(t) &:= e^{\lambda t}(t\underline{v}_1 + \underline{v}_2) \\ &= e^{-9t}\left(t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}\right)\end{aligned}$$

compare/ notice pattern

$$\begin{aligned}\mathbf{X}_3(t) &= e^{\lambda t}\left(\frac{t^2}{2!}\underline{v}_1 + t\underline{v}_2 + \underline{v}_3\right) \\ &= e^{-9t}\left(\frac{t^2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)\end{aligned}$$

General solution is
$$\mathbf{X}(t) = c_1\mathbf{X}_1(t) + c_2\mathbf{X}_2(t) + c_3\mathbf{X}_3(t).$$

If we needed $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4 \neq 0$ (in a different problem)

such that $(\underline{A} - \lambda I)^4 \underline{v}_4 = 0$

and chain
$$\begin{cases} (\underline{A} - \lambda I)\underline{v}_4 = \underline{v}_3 \\ (\underline{A} - \lambda I)\underline{v}_3 = \underline{v}_2 \\ (\underline{A} - \lambda I)\underline{v}_2 = \underline{v}_1 \\ (\underline{A} - \lambda I)\underline{v}_1 = 0 \end{cases},$$

then we'd have everything like before, but also add-in

$$\underline{X}_4(t) = e^{\lambda t}\left(\frac{t^3}{3!}\underline{v}_1 + \frac{t^2}{2!}\underline{v}_2 + t\underline{v}_3 + \underline{v}_4\right)$$

However, these are lengthy and don't show up in problems too much.