Sec 4.~~4~~, 4.5 Dimension & Column space

Remember: a basis for V (or W) is a linearly indep spanning set for V (or W)

Def: The dimension of vector space V (or subspace W) is the exact # of lin. indep vectors required to make a basis for V (or W).

- $\dim(\mathbb{R}^2) = 2$, $\cdot \dim(\mathbb{R}^3) = 3$, ...

"Theorem": The dimension of V (or W) is "unique": All bases ~~Any basis~~ of V (or W) must have the same number of vectors.

Ex: $\text{Null}\left(\begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix}\right)$ had a basis

$$\left\{ \underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{\underline{u}}, \underbrace{\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\underline{v}} \right\}$$

($\text{null}(\underline{A}) = \text{span}\{\underline{u}, \underline{v}\}$ & $\{\underline{u}, \underline{v}\}$ lin. indep)

so $\dim(\text{null}(\underline{A})) = 2$.

Same $\underline{A}$, ... can $\left\{ \begin{bmatrix} 5 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ be a basis for $\text{null}(\underline{A})$? No, it has too many vectors.

<u>Notes</u>: Say $\dim(V)$ is known, and $S = \{\underline{v}_1, \dots, \underline{v}_k\}$ is a set of vectors for V.

- If $k < \dim(V)$, then S cannot possibly span all of V. (LI/LD unknown)
- If $k > \dim(V)$, then S cannot be linearly indep. (may or may not span)
- If $k = \dim(V)$, it's not clear a-priori about span/indep.

★ Application: If $\underline{u}, \underline{v}, \underline{w}$ are vectors from $\mathbb{R}^3$, and they are lin. indep. Does $\{\underline{u}, \underline{v}, \underline{w}\}$ span all of $\mathbb{R}^3$?

Can there be some $\underline{x} \in \mathbb{R}^3$ such that

$\longrightarrow$ $\underline{x} \notin \text{span}\{\underline{u}, \underline{v}, \underline{w}\}$? (same as saying $\underline{x}$ is LI from $\{\underline{u}, \underline{v}, \underline{w}\}$)

If there was, then $\{\underline{u}, \underline{v}, \underline{w}, \underline{x}\}$ would be an LI set that would make a basis of length 4 for $\mathbb{R}^3$.

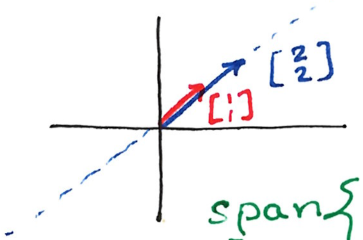
- <u>Column Space</u>: If $\underline{A} = [\underline{c}_1 \cdots \underline{c}_k]_{(n \times k)}$, then column space (of $\underline{A}$) is the span of its columns,

$$\text{col}(\underline{A}) \overset{\text{DEF}}{=} \text{span}\{\underline{c}_1, \dots, \underline{c}_k\}$$

Ex: $\underline{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\text{col}(\underline{A}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$

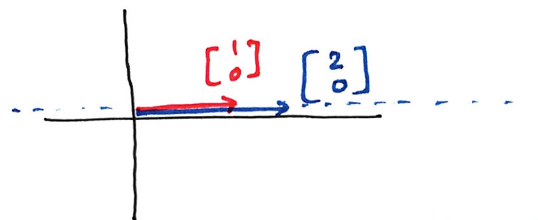
Note: Doing row ops to a matrix changes its column space:

$$\underline{A} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \longrightarrow \underline{B} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$



$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

$\text{col}(\underline{A})$



$\text{col}(\underline{B}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Read*: "How to find row/column space."