Basic Diff Eqs

1. (**Solutions**). Solutions can be
   - (*general/particular*): If they (do/don't) have unknown constant(s) in them.
   - (*explicit/implicit*): Explicit solutions are written "$y = \ldots$". Implicit solutions are written "$F(x, y) = C$" (or $F(x, y) = 3$, etc.).
2. (**Slope field**). Given ODE $y' = f(x, y)$, you can draw a slope field (and/or "check" a given one(s)) by plugging in a handful of test points (a, b) in the xy-plane and drawing a short, sloped line at that point.
3. (**Separable Eq**): Equation where you can write it as $g(y)dy = f(x)dx$.
   - Solve by just integrating both sides.
4. (**First-order Linear**): <u>Standard Form:</u> $y' + Py = Q$, where P,Q can depend on x.
   - Define the "integrating factor"
   $$\rho = e^{\int P\,dx}.$$
   - Then solve for y via
   $$\rho y = \int \rho Q\,dx + C.$$
5. (**Mixture Problem**): Always start with equations
   $$\frac{dA}{dt} = r_{\text{in}} c_{\text{in}} - r_{\text{out}} \frac{A(t)}{V(t)},$$
   $$V(t) = V_0 + (r_{\text{in}} - r_{\text{out}})t.$$
   - Note that essentially c_{out} was replaced by $\frac{A(t)}{V(t)}$.
   - This is a first-order linear eq. Put into *standard form* to solve for $A(t)$.

Substitutions

6. $y' = f(ax + by + c)$. (This type doesn't have a name).
   $$v = ax + by + c$$
   - Equation becomes *separable*.
7. (**Homogeneous**): Equation where $(x \to cx, y \to cy)$ has no effect after simplifying.
   $$v = y/x$$
   - Equation becomes *separable*.
8. (**Bernoulli**): $y' + Py = Qy^n$,
   $$v = y^{1-n}$$
   - Equation becomes *first-order linear*.
   - **Pro Tip:** The first-order linear equation you should get after the sub is:
   $$v' + \big((1-n)P\big)v = (1-n)Q$$

9. (**Second-order**): First, always use

$$v = y' = \frac{dy}{dx} \qquad \text{and} \qquad y'' = \frac{dv}{dx}.$$

Then, you have two options for what to do with $y'' = \frac{dv}{dx}$. You can either:

(1) Keep this $y'' = \frac{dv}{dx}$.

(2) Rewrite this as $y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$.

Use whichever choice of $y'' = \frac{dv}{dx}$ or $y'' = v\frac{dv}{dy}$ leaves your ODE with just two variables.

- Equation can become either *separable* or *first-order linear*.

Exact Eqs; Population and Stability

10. (**Exact Eq**): Equations $Mdx + Ndy = 0$ and/or $\frac{dy}{dx} = -\frac{M}{N}$ are exact when

$$M_y = N_x.$$

11. (**Logisitic Eq**): Equation $\frac{dP}{dt} = kP(M-P)$, where k and M are constants and > 0.

- M is the *limiting population.*
- Equation is separable; solve using partial fractions.
- If $P(0) = P_0$, then particular solution is

$$P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-kMt}}.$$

12. (**Autonomous Eq**): Equation of form $\frac{dy}{dt} = f(y)$ or $\frac{dx}{dt} = f(x)$; thus, no independent variable on right-hand side.

13. (**Critical points**): Given autonomous eq $y' = f(y)$ (or $x' = f(x)$), the values $y = c$ such that $y' = f(c) = 0$ are the *critical points* (or $x' = 0$, etc.).

- Critical points can be *stable*, *semistable*, or *unstable.*
- If $y = c$ is a critical point, then the constant function $y(x) \equiv c$ (or $y(t) \equiv c$, or $x(t) \equiv c$, etc.) is an *equilibrium solution.*

Systems of linear eqs

Definition (**Consistent** vs. **Inconsistent**). A system of linear eqs can have three things happen regarding its solutions. Specifically, the system can have:

1. No solutions;
2. A single/unique solution;
3. Infinitely many solutions.

If the system has *no solutions*, we say the system is inconsistent. If the system has either *a unique solution* or *infinitely many solutions*, then we call the system consistent.

Definition. In any matrix, the first *nonzero* entry of a row is called a pivot. If a row has a pivot then it is a "pivot row", and if a column has a pivot it is a "pivot column". Thus, a row of all 0's isn't a pivot row.

Definition (**(Reduced) Echelon Form**)**.** There are four basic conditions:

- (i) Entries *beneath* pivots are all 0;
- (ii) Rows of 0's (if there are any) moved to bottom;
- (iii) Entries *above* pivots are all 0;
- (iv) All pivot entries are equal to 1.

- If (at least) (i) and (ii) hold, then matrix is in echelon form (EF).
- If (i), (ii), (iii), and (iv) *all* hold, then matrix is in reduced echelon form (REF).

Note: Any matrix "has" (or "can-be-reduced-to") an (EF), not just augmented matrices. A matrix can have many different (EF)'s, but it will only have one (REF).

Definition (**Row Operations**)**.** In reducing matrices to (EF) or (REF) there are three *row operations* we can use:

1. Swap two rows: $(\mathrm{R}_i \leftrightarrow \mathrm{R}_j)$

$$(\mathrm{R}_1 \leftrightarrow \mathrm{R}_3): \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix};$$

2. Scale a row: $(\mathrm{R}_i \to c\mathrm{R}_i)$

$$(\mathrm{R}_3 \to \tfrac{1}{2}\mathrm{R}_3): \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 14 & 16 & 18 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix};$$

3. Add a (scaled-)copy of one row to another: $(\mathrm{R}_i \to \mathrm{R}_i + c\mathrm{R}_j)$

$$(\mathrm{R}_3 \to \mathrm{R}_3 - 7\mathrm{R}_1): \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & -6 & -12 \end{bmatrix}.$$

Note: The matrices I put above are all just "plain" matrices—they are not "augmented" matrices b/c we aren't solving a system of linear eqs here.

How to solve a system of linear eqs.

To solve a system of linear eqs, we *always* use the first 3 steps:

1. Write the *augmented matrix* of the system.
2. Use row operations to reduce (*working "top down") the augmented matrix to echelon form (EF).
3. Use the (EF) to identify which variables are pivot variables (variables with pivots in their column), and which are free variables (variables without).

   (3.5) Assign each free variable a "free parameter" letter. Though it doesn't matter which letters you use, we usually use s, t, and u (if necessary).

Then we have a choice for the last “step”:

(4A) From (EF), translate back to equations, and work *bottom up* “substituting” variables to get equations

$$(\textit{pivot vars}) = (\#) + (\textit{free variable(s), if any}).$$

(4B) From (EF), continue using row operations until the matrix is in <u>reduced</u> echeclon form (REF). Then translate back to equations and “read off” equations

$$(\textit{pivot vars}) = (\#) + (\textit{free variable(s), if any}).$$

- If we use (1)–(3) and then (4A), this method is called “*Gaussian elimination*”, or just “*the method of elimination*”.
- If we use (1)–(3) and then (4B), this method is “*Gauss-Jordan elimination*”.
- Both methods take roughly equal amounts of work/time, though I prefer reducing to (REF) and not worrying about substitution.

Facts/Tips/Tricks.

1. A system will be *inconsistent* (aka have no solutions) if the augmented matrix of the system has <u>any</u> (EF) (doesn't have to be (REF)) that has a row that looks like

$$\left[\begin{array}{cc|c} 0 & 0 & (\neq 0) \end{array}\right], \quad \left[\begin{array}{ccc|c} 0 & 0 & 0 & (\neq 0) \end{array}\right], \quad \text{etc.},$$

where “($\neq 0$)” just means any number that isn't 0. This is the same as saying that the last column <u>of the augmented matrix</u> has a pivot. (Emphasis b/c if it's not an augmented matrix of a system, i.e. it's just a plain matrix, then we aren't concerned with solutions).

2. A system has a *unique solution* if: The system is consistent, and in the augmented matrix of the system, every variable is a pivot variable. That is:

$$(\text{consistent}) + (\text{no free vars}) \Longrightarrow (\text{unique solution}).$$

3. If a system has *infinitely many solutions* if: System is consistent (so it has at least one solution) and it has <u>any</u> free variables. That is:

$$(\text{consistent}) + (\text{at least one free var}) \Longrightarrow (\text{infinitely many solutions}).$$

4. It's good to start with a “1” as your top pivot when reducing a matrix, so swap rows and/or scale them to simplify things if necessary. For example, starting with the matrix

$$\begin{bmatrix} 3 & 8 & 7 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix}$$

and trying to reduce/eliminate with

$$(\mathrm{R}_2 \to \mathrm{R}_2 - \tfrac{1}{3}\mathrm{R}_1): \quad \begin{bmatrix} 3 & 8 & 7 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 8 & 7 \\ 0 & -\frac{2}{3} & -\frac{4}{3} \\ 2 & 7 & 9 \end{bmatrix}$$

is messier than first Swapping, then Adding:

$$(\mathrm{R}_1 \leftrightarrow \mathrm{R}_2): \quad \begin{bmatrix} 3 & 8 & 7 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 7 \\ 2 & 7 & 9 \end{bmatrix}$$

$$(\mathrm{R}_2 \rightarrow \mathrm{R}_2 - 3\mathrm{R}_1): \quad \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 7 \\ 2 & 7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 4 \\ 2 & 7 & 9 \end{bmatrix}.$$

You <u>will</u> get the same answers/solutions either way (especially because you'll eventually end up at the <u>one</u> (REF)), but the path to get there can be messier/cleaner.