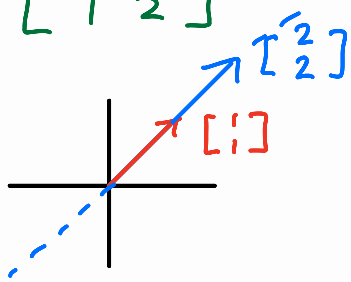
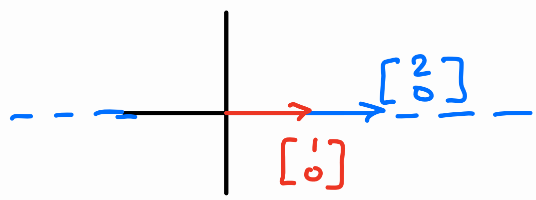
Column space $\text{Col}(\underline{A})$ = span of columns of $\underline{A}$

Ex: $\text{col}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$

Note: row ops modify column space:

$$\underset{\underline{A}}{\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}} \longrightarrow \underset{\underline{B}}{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}}$$





$\text{col}(\underline{A}) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ $\qquad$ $\text{col}(\underline{B}) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

How to find a basis for $\text{Col}(\underline{A})$

Ex: $$\underline{A} = \begin{array}{c} \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix} \\ \quad \underline{v_1} \quad\; \underline{v_2} \quad\; \underline{v_3} \quad\; \underline{v_4} \end{array}$$

$\text{col}(\underline{A}) = \text{span}\{\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}\}$.

- Now, $\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}$ are 4 vectors from $\mathbb{R}^3$; since we have more vectors than $\dim(\mathbb{R}^3)$, $\{\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}\}$ must be LD, so they aren't a basis for $\text{col}(\underline{A})$.
- To make a basis then, we want to discard some of $\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}$ and keep some linearly indep subset of $\{\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}\}$ ... that LI set will be our basis of $\text{col}(\underline{A})$.

• To do this ↑ we use the following method:

1) Reduce $\underline{A}$ to echelon form* using row ops

$$\underline{A} \to \cdots \to \underbrace{\begin{bmatrix} \boxed{1} & -4 & -3 & -7 \\ 0 & \boxed{1} & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\underline{B}}$$

* REF is okay, but we can stop after EF when finding a basis for col($\underline{A}$).

2) locate pivot columns of $\underline{B}$ (col 1, col 2)

3) Use the corresponding columns of $\underline{A}$ NOT $\underline{B}$ as the basis for Col($\underline{A}$)

col 1, col 2 pivots, so use $\underline{v_1}$, $\underline{v_2}$ from $\underline{A}$

$$\text{Col}(\underline{A}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} \right\}$$

★ Note: Though not obvious from the method, the vectors we get from this method will always be LI (not just in this example, but in general) so they are indeed a basis for col($\underline{A}$).

If you are interested in "why" this works, I will happily explain in a later note and before/as we review for Exam 2. At present I want to avoid overwhelming you with info.

Def: (column) rank of $\underline{A}$: (aka the "rank of $\underline{A}$")

$$\text{rank}(\underline{A}) = \dim(\text{col}(\underline{A}))$$

$$\left(\begin{aligned} &= \#\text{ of pivot cols of (EF of } \underline{A}) \\ &= \#\text{ of pivots " " " } \end{aligned} \right)$$

Row Space

Row space $\overset{\text{DEF}}{=}$ span of row vectors of $\underline{A}$,

denoted as row($\underline{A}$).

Ex: $\underline{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow \text{row}(\underline{A}) = \text{span}\{[1\ 2], [3, 4]\}$

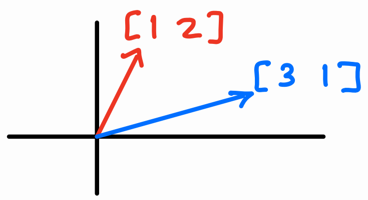
Ex: $\text{row}\left(\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\{[2\ 2], [0\ 0]\}$

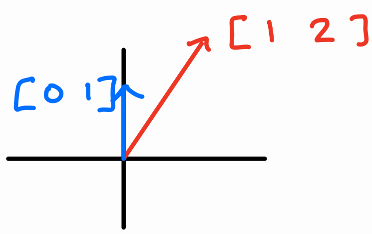
$$= \text{span}\{[2\ 2]\}$$

Visualizing row vectors and row space

- Can "plot" row vectors in $\mathbb{R}^n$ just like col. vecs.
- Independence, basis, etc. all work the same for lin. comb. of row vectors.

Ex: $\underset{\underline{A}}{\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \rightarrow \underset{\underline{B}}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} \longrightarrow$

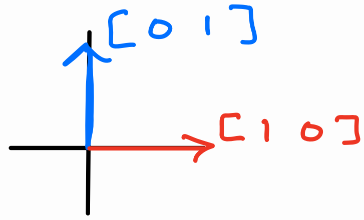




$\text{row}(\underline{A}) = \text{span}\{[1\ 2], [3\ 1]\} = \mathbb{R}^2$

$\text{row}(\underline{B}) = \text{span}\{[0\ 1], [1\ 2]\} = \mathbb{R}^2$

$$\to \quad \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{\underline{C}}$$



$row(\underline{C}) = span\{[1\ 0], [0\ 1]\} = \mathbb{R}^2$

- Notice that here $row(\underline{A}) = row(\underline{B}) = row(\underline{C})$

  (they all $= \mathbb{R}^2$)

Thm: row ops do not change row space of matrix.
(OTOH, row ops do change column space)

Def: If $\underline{A}$ can be changed into $\underline{B}$ via row ops, we say $\underline{A}$ and $\underline{B}$ are "row equivalent".

Thus, the theorem says that "if $\underline{A}$ and $\underline{B}$ are row equiv. then $row(\underline{A}) = row(\underline{B})$"

How to compute a basis for the row space of $\underline{A}$?

Ex: $\underline{A} = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}$

(method is very similar to col(A) method)

1) Reduce $\underline{A}$ to echelon form using row ops
   (again "plain" EF is fine)

$$\underline{A} \to \begin{pmatrix} 3 \text{ row} \\ \text{ops} \end{pmatrix} \to \underbrace{\begin{bmatrix} \boxed{1} & -4 & -3 & -7 \\ 0 & \boxed{1} & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{"}B\text{"}}$$

2) locate pivot rows of $\underline{B}$ (row 1, row 2)

3) the pivot rows of $\underline{B}$ make a basis for row($\underline{A}$).

$$\text{row}(\underline{A}) = \text{row}(\underline{B}) = \text{span}\{\underbrace{[1\ -4\ -3\ -7], [0\ 1\ 1\ 3]}_{\text{basis for row}(\underline{A})}\}$$

★ Notice when computing basis for col($\underline{A}$) we used the columns of the original $\underline{A}$ as the basis, but we used the rows of the reduced $\underline{B}$ as the basis for row($\underline{A}$).

__Def__: The (row) __rank__ of $\underline{A}$:

$$\text{rank}(\underline{A}) \overset{\text{DEF}}{=} \text{dimen}(\text{row}(\underline{A}))$$

$$(\text{rank}(\underline{A}) = 2 \text{ above})$$

$$\begin{aligned} \text{row rank} &= (\#\text{ of pivot rows of } (\text{EF of } \underline{A})) \\ &= (\#\text{ of pivots of } (\text{EF of } \underline{A})) \end{aligned}$$

__Thm__: The row rank of $\underline{A}$ is always equal to the column rank of $\underline{A}$.

(b/c # pivot rows of (EF of $\underline{A}$)
= # pivot cols " " "
= # pivots " " " ")