Sec 3.2 – 3.3 Matrices and Elimination (cont.)

Last time: Echelon form:

pivots

$$\left[\begin{array}{ccccc|c} \boxed{1} & 2 & 3 & 4 & 5 & -1 \\ 0 & \boxed{2} & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & \boxed{-1} & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad (4\times 6)$$

row of 0's at bottom

entries under pivot are 0

"(Elementary) Row Operations":

1) Swap rows $(R_i \longleftrightarrow R_j)$
2) Scale a row $(R_i \longrightarrow c\cdot R_i)$
3) Add copy of row $(R_i \longrightarrow R_i + c\cdot R_j)$

Method of (Gaussian) Elimination

I) "Eliminate" variables top-down using row ops, until matrix is in echelon form.

II) "Substitute" variables "bottom-up"

~~Met~~ **Reduced echelon form (REF):**

1) Needs to already be in (EF)
2) all pivots scaled to 1 (use Scale if needed)
3) Entries above and below pivot are 0.

$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 0 & 22 & 11 \\ 0 & \boxed{1} & \frac{3}{2} & 0 & -5/2 & -2 \\ 0 & 0 & 0 & \boxed{1} & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Sec 3.2 Matrices & Elimination

Matrix: Array with rows and columns,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad (2 \times 2) \qquad \begin{bmatrix} 1 & 2 & 3 \\ 6 & 2 & 1 \end{bmatrix} \quad (2 \times 3)$$

rows × columns

a column vector is an $m \times 1$ matrix $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

an alternative notation is tuples (x_1, x_2, x_3) for column vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_1, x_2, x_3)$

a row vector is ~~a~~ $1 \times n$ matrix $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$

Matrices simply provide a shorthand for manipulating and solving systems of linear equations.

$$\begin{cases} x + 3y = 9 & (R_1) \\ 2x + y = 8 & (R_2) \end{cases} \longleftrightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & 3 & 9 \\ 2 & 1 & 8 \end{array}\right] \end{array}$$

★ "Eliminate variables top → bottom"

$$(R_2 \to R_2 - 2R_1) \begin{cases} x + 3y = 9 \\ 0 + \underbrace{-5y = -10} \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 3 & 9 \\ 0 & -5 & -10 \end{array}\right]$$

"x eliminated" $\Rightarrow \boxed{y = 2}$

Now we "back substitute" our value for y "bottom up"

(R_1) becomes $x + 6 = 9 \Rightarrow \boxed{x = 3}$

So solution to system is $(x, y) = (3, 2)$, equiv. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

vector form

Look at $\left[\begin{array}{cc|c} \boxed{1} & 3 & 9 \\ 0 & \boxed{-5} & -10 \end{array}\right]$

Defn: The first nonzero entry in a row is a "pivot" (the location)

Defn: a matrix is in (row-) echelon form (EF) if:

1) Entries below all pivot(s) are 0;
2) any rows of all 0's are moved to bottom.

Ex $\begin{cases} 3x - 8y + 10z = 22 \\ x - 3y + 2z = 5 \\ 2x - 9y - 8z = -11 \end{cases}$

$$\begin{array}{cccc} x & y & z & \#'s \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 3 & -8 & 10 & 22 \\ 1 & -3 & 2 & 5 \\ 2 & -9 & -8 & -11 \end{array}\right]$$

augmented matrix of the system

coefficient matrix of system

$(R_2 \longleftrightarrow R_1)$ $\left[\begin{array}{ccc|c} \boxed{1} & -3 & 2 & 5 \\ \boxed{3} & -8 & 10 & 22 \\ \boxed{2} & -9 & -8 & -11 \end{array}\right]$

Now "eliminate x" in R_2 and R_3

$\begin{pmatrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix}$ $\left[\begin{array}{ccc|c} \boxed{1} & -3 & 2 & 5 \\ 0 & \boxed{1} & 4 & 7 \\ 0 & \boxed{-3} & -12 & -21 \end{array}\right]$

Now eliminate y from (R_3)

$$(R_3 \to R_3 + 3R_2) \quad \begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} \boxed{1} & -3 & 2 & 5 \\ 0 & \boxed{1} & 4 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (EF)$$

x, y are pivot variables

no pivot in z column, so we say z is a free variable

("$z \in \mathbb{R}$")

"in"

$x = ?$
$y = ?$
$z = ?$

$$\begin{cases} x - 3y + 2z = 5 \\ y + 4z = 7 \\ 0 = 0 \end{cases}$$

"Substitute bottom-up"

- z free

$(R_2) \implies y = 7 - 4z$

$(R_1) \implies x = 5 + 3(7-4z) - 2z$

$x = 5 + 21 - 12z - 2z$

$x = 26 - 14z$

Solutions are
$$\begin{cases} x = 26 - 14z \\ y = 7 - 4z \\ z \text{ free} \end{cases}$$

Often for emphasis, when z (or other) is a free variable, we let $t \in \mathbb{R}$ be a free parameter, and set $z = t$.

So we write solutions as

$$\begin{cases} x = 26 - 14t \\ y = 7 - 4t \\ z = t \end{cases}, \quad t \text{ free}$$

Solution in vector form:

$$(x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 26 - 14t \\ 7 - 4t \\ t \end{bmatrix} = \begin{bmatrix} 26 \\ 7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -14 \\ -4 \\ 1 \end{bmatrix}$$

Ex $\begin{cases} x + 2y + z = 4 \\ 3x + 8y + 7z = 20 \\ 2x + 7y + 9z = 23 \end{cases}$ (can use x_1, x_2, x_3 in place of x, y, z)

I) Matrix to (EF):

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 4 \\ \boxed{3} & 8 & 7 & 20 \\ \boxed{2} & 7 & 9 & 23 \end{array}\right]$$

$\begin{pmatrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix}$ $\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 4 \\ 0 & \boxed{\cancel{2}}\,1 & \cancel{4}\,2 & \cancel{8}\,4 \\ 0 & \boxed{3} & 7 & 15 \end{array}\right]$

after, do $(R_2 \to \frac{1}{2}R_2)$

$(R_3 \to R_3 - 3R_2)$ $\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 4 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right]$ (EF)

x, y, z all pivot variables.

II) Substitution "bottom-up"

$$\begin{cases} x + 2y + z = 4 \\ y + 2z = 4 \\ \boxed{z = 3} \end{cases}$$

$R_2 \implies y = 4 - 2z = \underline{-2}$

$R_1 \implies x = 4 - 2y - z = 4 + 4 - 3 = \underline{5}$

Solution(s) to system are $\begin{cases} x = 5 \\ y = -2 \\ z = 3 \end{cases}$ (single/unique solution)

<u>vector form</u> : $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$

- any free variables $\implies$ ∞-ly many solutions.
- all pivot <u>variables</u> $\implies$ single/unique solution
- Pivot in last (bar) column $\left(\text{ex} : \left[\begin{array}{cc|c} \cdots & & \cdots \\ 0 & 0 & \boxed{1} \end{array}\right] \right)$ ← $0=1$

  $\implies$ no solutions.

Ex

$$\begin{array}{c} \left[\begin{array}{ccccc|c} \boxed{1} & 2 & 3 & 4 & 5 & -1 \\ 0 & \boxed{2} & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & \boxed{-1} & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array} \quad (EF)$$

- x_1, x_2, x_4 pivot variables.
- x_3, x_5 free variables.

Eqs are
$$\begin{cases} x_1 = -1 - 2x_2 - 3x_3 - 4x_4 - 5x_5 \\ 2x_2 = 0 - 3x_3 + 2x_4 - x_5 \\ -x_4 = 2 - 3x_5 \\ 0 = 0 \end{cases}$$

x_3, x_5 are free, so let s, t be free params,

set $\begin{cases} x_3 = s \\ x_5 = t \end{cases}$

"substitute bottom up":

$$x_4 = -2 + 3t$$

... continue Friday...