Ex (Echelon form)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= -1 \\ 2x_2 + 3x_3 - 2x_4 + x_5 &= 0 \\ -1x_4 + 3x_5 &= 2 \end{aligned}$$

$$\overbrace{\left[\begin{array}{ccccc|c} \boxed{1} & 2 & 3 & 4 & 5 & -1 \\ 0 & \boxed{2} & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & \boxed{-1} & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}^{\text{augmented matrix}} \quad (EF)$$

coefficient matrix

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

- Because augmented matrix is in (EF), we can solve system using "back-substitution":

x_1, x_2, x_4 pivot

x_3, x_5 free

As usual, let s, t be free parameters, $(s, t \in \mathbb{R})$,

set $\begin{cases} x_3 = s \\ x_5 = t \end{cases}$,

solve for pivots x_1, x_2, x_4:

$$\begin{cases} x_1 = -1 - 2x_2 - 3x_3 - 4x_4 - 5x_5 \\ x_2 = -\frac{3}{2}x_3 + x_4 - \frac{1}{2}x_5 \\ x_4 = -2 + 3x_5 \end{cases}$$

$\hookrightarrow$ $\underline{x_4 = -2 + 3t}$

$x_2 = -\frac{3}{2}s + (-2 + 3t) - \frac{1}{2}t$

$\underline{x_2 = -2 - \frac{3}{2}s + \frac{5}{2}t}$

$x_1 = -1 - 2\left(-2 - \frac{3}{2}s + \frac{5}{2}t\right) - 3s - 4(-2 + 3t) - 5t$

$= \underline{-1 + 4} + \underline{3s} \underline{- 5t} \underline{- 3s} + \underline{8} \underline{- 12t - 5t}$

$\underline{x_1 = 11 + 0 - 22t}$

solutions are "scalar form"

$$\begin{cases} x_1 = 11 - 22t \\ x_2 = -2 - \frac{3}{2}s + \frac{5}{2}t \\ x_3 = s \\ x_4 = -2 + 3t \\ x_5 = t \end{cases}, \quad \underline{s, t \text{ free}}$$

vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 11 - 22t \\ -2 - \frac{3}{2}s + \frac{5}{2}t \\ s \\ -2 + 3t \\ t \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ -2 \\ 0 \\ -2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -22 \\ 5/2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

<u>Reduced Echelon Form</u> : "reduce/eliminate more, and then "read off solution"

<u>Ex</u> :

$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ 3x_1 + 8x_2 + 7x_3 = 20 \\ 2x_1 + 7x_2 + 9x_3 = 23 \end{cases}$$

... Echelon Form (augmented matrix) was

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 4 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right] \qquad \begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_2 + 2x_3 &= 4 \\ x_3 &= 3 \end{aligned}$$

$$\begin{pmatrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - 2R_3 \end{pmatrix} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 1 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right] \qquad \begin{aligned} x_1 + 2x_2 &= 1 \\ x_2 &= -2 \\ x_3 &= 3 \end{aligned}$$

$$(R_1 \to R_1 - 2R_2) \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right] \longleftrightarrow \begin{cases} x_1 = 5 \\ x_2 = -2 \\ x_3 = 3 \end{cases}$$

Ex
$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ 3x_1 + 8x_2 + 7x_3 = 20 \\ 2x_1 + 7x_2 + 9x_3 = 23 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right]$$ aug. matrix.

$$\xrightarrow{\text{row ops}} \cdots \longrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 4 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right] \qquad \begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_2 + 2x_3 &= 4 \\ x_3 &= 3 \end{aligned}$$

$$\begin{pmatrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - 2R_3 \end{pmatrix} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 1 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right] \qquad \begin{aligned} x_1 + 2x_2 &= 1 \\ x_2 &= -2 \\ x_3 &= 3 \end{aligned}$$

$$(R_1 \to R_1 - 2R_2) \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right] \qquad \begin{aligned} x_1 &= 5 \\ x_2 &= -2 \\ x_3 &= 3 \end{aligned}$$

(REF)

★ Every matrix can be put into (EF) using row ops, but there can be many different (EF) that you can "get to". However, every matrix has a single, unique (REF).

- Given a system of linear eqs; to solve we can start w/ augmented matrix, then:
  1) Put aug. mat. into (EF), then substitute ("Gaussian elimination")
  2) Put aug. mat. into (REF), then "read off" solution (Gauss–Jordan elimination)