What are row operations? Let's note/see two ways to multiply matrices:

$$\underline{A} = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Method 1 : ("column weights") This is the version we used in class.

$\underline{AB}$: $(3\times\underbrace{3)(3}_{\checkmark}\times 3) \longrightarrow$ result is 3×3.

$$\underline{AB} = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

column 1 of $\underline{AB}$: $1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 3 \\ 13 \end{bmatrix}$

Column 2 of $\underline{AB}$: $2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ -5 \end{bmatrix}$

column 3 of $\underline{AB}$: $1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix}$

so $\underline{AB} = \begin{bmatrix} 17 & -4 & 6 \\ 3 & -2 & 0 \\ 13 & -5 & 4 \end{bmatrix}$

This method is good for visualizing combinations of vectors in space, and visualizing solutions for vector equations $\underline{Ax} = \underline{b}$

Method 2 ("row weights") Previously we used $\underline{B}$ columns to weight $\underline{A}$ columns. Alternatively we can use $\underline{A}$-rows to weight $\underline{B}$-columns

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

row 1 of $\underline{AB}$:
$$1\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + 2\begin{bmatrix} 3 & -3 & 0 \end{bmatrix} + 5\begin{bmatrix} 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 17 & -4 & 6 \end{bmatrix}$$

row 2 of $\underline{AB}$:
$$-1\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + 0\begin{bmatrix} 3 & -3 & 0 \end{bmatrix} + 2\begin{bmatrix} 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 \end{bmatrix}$$

row 3 of $\underline{AB}$:
$$2\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + 3\begin{bmatrix} 3 & -3 & 0 \end{bmatrix} + 1\begin{bmatrix} 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 13 & -5 & 4 \end{bmatrix}$$

Thus, again $\underline{AB} = \begin{bmatrix} 17 & -4 & 6 \\ 3 & -2 & 0 \\ 13 & -5 & 4 \end{bmatrix}$

The result $\underline{AB}$ is the same either way, but this "row weights" version ties-in quite a lot with our row operations we use.

Elementary row operations (by examples)

1) Swapping Rows : (Swap $R_2 \longleftrightarrow R_3$)

- (new R_1) $= 1$(old R_1)
- (new R_2) is $= 1$(old R_3)
- (new R_3) $= 1$(old R_2)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix} \quad \leftarrow \text{no change}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 9 \\ 3 & -3 & 0 \end{bmatrix}$$

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2) Scaling a row by c : (ex: $R_2 \to \frac{1}{3} R_2$)

(new R_2) $= \frac{1}{3}$(old R_2) $\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 2 & 7 & 9 \end{bmatrix}$$

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3) Adding multiple of a row (Ex: $R_3 \to R_3 - 2R_1$)

(new R_3) $=$
-2(old R_1) $+ 1$(old R_3) $\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 0 & 3 & 7 \end{bmatrix}$$

Thus, changing into <u>echelon form or reduced echelon form</u> is accomplished by chaining together multiplications of the types above:

<u>For example</u>:

<u>Reduce $\underline{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix}$ to (REF)</u>

$\begin{pmatrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & -3 & 0 \\ 2 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -9 & -3 \\ 0 & 3 & 7 \end{bmatrix},$$

$\left(R_2 \to -\frac{1}{9} R_2\right)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/9 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & -9 & -3 \\ 0 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/3 \\ 0 & 3 & 7 \end{bmatrix},$$

$(R_3 \to R_3 - 3R_2)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1/3 \\ 0 & 3 & 7 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{1} & 1/3 \\ 0 & 0 & \boxed{6} \end{bmatrix} \quad \text{(EF)}$$

So, in total, we see that

$$\left(\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}}_{\underline{E_3}}\underbrace{\begin{bmatrix}1&0&0\\0&-1/9&0\\0&0&1\end{bmatrix}}_{\underline{E_2}}\underbrace{\begin{bmatrix}1&0&0\\-3&1&0\\-2&0&1\end{bmatrix}}_{\underline{E_1}}\right)\underbrace{\begin{bmatrix}1&2&1\\3&-3&0\\2&7&9\end{bmatrix}}_{\underline{A}}$$

$$= \begin{bmatrix}1&0&0\\-3&-1/9&0\\-2&-3&1\end{bmatrix}\underline{A} = \begin{bmatrix}1&2&1\\0&1&1/3\\0&0&6\end{bmatrix},$$

So multiplying $\underline{A}$ by this single matrix changes $\underline{A}$ to (EF). Similarly, we could go further and find a single matrix to change $\underline{A}$ into (REF)

A "third" method (but not really different)

$$\underbrace{\begin{bmatrix}1&2&5\\-1&0&2\\2&3&-7\end{bmatrix}}_{\underline{A}}\underbrace{\begin{bmatrix}1&2&1\\3&-3&0\\2&7&9\end{bmatrix}}_{\underline{B}} = \begin{bmatrix}&31&\\&&17\\&&\end{bmatrix}$$

"entrywise": the (i,j)-entry of $\underline{AB}$ is computed by pairing elements of <u>row i of $\underline{A}$</u> and <u>column j of $\underline{B}$</u>

<u>Ex:</u> $(1,2)$ entry $= 1(2) + 2(-3) + 5(7) = 31$

$(2,3)$ entry $= -1(1) + 0(0) + 2(9) = 17$

and so on ...