Sec 3.4 Matrix Operations

★ Matrices and vectors are always underlined!

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad a_{\underset{\text{row}}{i}\underset{\text{column}}{j}} \text{ are the "entries/elements" of } \underline{A}$$

" a_{21} is the $(2,1)$-entry of $\underline{A}$ "

Scalar Multiplication (Scalar: number) :
$$3\underline{A} = \begin{bmatrix} 3a_{11} & 3a_{12} \\ 3a_{21} & 3a_{22} \end{bmatrix}$$

Addition: · done "elementwise"

· **Dimensions** (#rows, # cols) must match between matrices.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{bmatrix}$$

Matrix multiplication: ★ NOT elementwise

- **Requirement**: For product of $\underline{A}\ \underline{B}$ to be defined, we must have (# cols of $\underline{A}$) = (# rows of $\underline{B}$)

If we have that, say $\underline{A}$ $(m \times n)$ and $\underline{B}$ $(n \times k)$

then $\underline{AB}$: $(m \times \underbrace{n)(n}_{\text{match}\checkmark} \times k)$ → result $\underline{AB}$ is $(m \times k)$

First idea: two vectors $\overset{a}{\cancel{\underline{y}}} = [\overset{a}{\cancel{y_1}}\ \ \overset{a}{\cancel{y_2}}\ \ \overset{a}{\cancel{y_3}}]_{(1\times 3)}$ $\quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{(3\times 1)}$

product $\underline{ax}$: $(1 \times \underbrace{3)(3}_{\checkmark} \times 1)$ result will be a 1×1 matrix, (which is just a scalar.)

$$\underline{ax} = [a_1 \ a_2 \ a_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \overset{DEF}{=} \underbrace{x_1 a_1 + x_2 a_2 + x_3 a_3}_{\text{like the dot product.}}$$

$\langle a_1, a_2, a_3 \rangle \quad \langle x_1, x_2, x_3 \rangle$

Let $\underline{v_1} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\underline{v_2} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\underline{v_3} = \begin{bmatrix} 7 \\ 1 \\ -3 \end{bmatrix}$ (all 3×1)

$$\underline{A} = [\underline{v_1} \ \underline{v_2} \ \underline{v_3}] \overset{DEF}{=} \begin{bmatrix} 1 & 2 & 7 \\ 3 & 0 & 1 \\ 5 & -1 & -3 \end{bmatrix} \ (3\times 3)$$

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\left(\text{equivalent to } \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} \right)$

Is $\underline{Ax}$ defined? $\underline{Ax}$: $(3\times 3)(3\times 1) \longrightarrow (3\times 1)$

($\underline{xA}$ not defined b/c of mismatch)

★ $$\underline{Ax} = [\underline{v_1} \ \underline{v_2} \ \underline{v_3}]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \overset{DEF}{=} x_1 \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3}$$

the x_i are like "weights" for $\underline{v_i}$

Ex $$\begin{array}{ccc} \underline{v_1} & \underline{v_2} & \underline{v_3} \end{array}$$
$$\begin{bmatrix} 1 & 2 & 7 \\ 3 & 0 & 1 \\ 5 & -1 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = 3\underline{v_1} + \underline{v_2} - \underline{v_3}$$
$$= \begin{bmatrix} 3 \\ 9 \\ 15 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -7 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 17 \end{bmatrix}$$

"vector eqs": $\underline{A} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$

$\qquad\qquad\quad \underline{v_1} \ \ \underline{v_2}$

Eq: $\underline{Ax} = \underline{b}$ same as $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$

$\qquad\qquad\qquad\qquad\qquad \underline{v_1} \ \ \underline{v_2}$

$x_1 \underline{v_1} + x_2 \underline{v_2}$ $\quad x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$

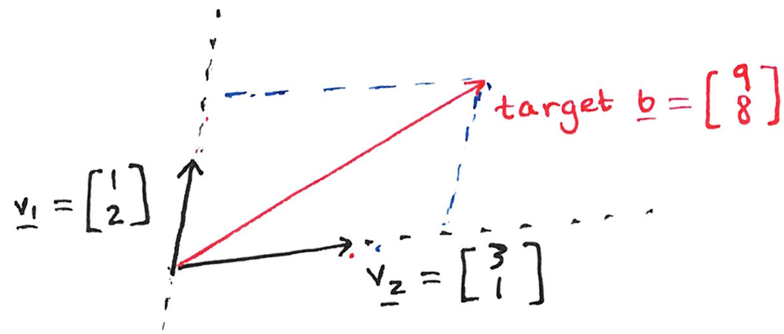
$$\begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} =$$

$$\begin{bmatrix} x_1 + 3x_2 \\ 2x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

★ ~~tot~~ finding $\underline{x}$ that solves $A\underline{x} = \underline{b}$ is the same as solving system

$$\begin{cases} x_1 + 3x_2 = 9 \\ 2x_1 + x_2 = 8 \end{cases}$$

solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$



Keep $\underline{A} = [\underline{v_1}\ \underline{v_2}\ \underline{v_3}]$, $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, now $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, and

let $\underline{B} = [\underline{x}\ \&\ \underline{y}]_{(3\times 2)}$

$\underline{AB}$ defined? $\quad (3\times 3)(3\times 2) \longrightarrow (3\times 2)$ ✓

$$\underline{AB} = \underline{A}\,[\underline{x}\ \ \underline{y}] \overset{DEF}{=} [\underline{Ax}\ \ \underline{Ay}]$$

Ex:
$$\begin{array}{cc} \begin{bmatrix} 1 & 2 & 7 \\ 3 & 0 & 1 \\ 5 & -1 & -3 \end{bmatrix} & \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ -1 & 3 \end{bmatrix} \\ \underline{v_1}\quad \underline{v_2}\quad \underline{v_3} & \underline{x}\quad \underline{y} \end{array}$$

column 1 of $\underline{AB}$ is $\underline{Ax} = \begin{bmatrix} -2 \\ 8 \\ 17 \end{bmatrix}$

column 2 of $\underline{AB}$ is $\underline{Ay} = 2\underline{v_1} + \underline{v_2} + 3\underline{v_3}$

$$= \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 21 \\ 3 \\ -9 \end{bmatrix}$$

$$= \begin{bmatrix} 25 \\ 9 \\ 0 \end{bmatrix}$$

$$\underline{AB} = \begin{bmatrix} -2 & 25 \\ 8 & 9 \\ 17 & 0 \end{bmatrix}.$$

Read note "what are row operations?"