# How to compute $\underline{A}^{-1}$ if possible. (Ex: 3x3)

## General method:

1) Start with $\underline{A}$ $n \times n$,
2) Make "augmented" matrix $\left[\underline{A} \,\middle|\, \underline{I_n}\right]$
3) Do row operations to reduce $\underline{A}$ to (REF).
   (note these affect augmented part too)
   At the end of this step you have a matrix
   $$\left[(\text{REF } \underline{A}) \,\middle|\, \underline{B}\right]$$
4) Two possibilities now:
   i) If $(\text{REF } \underline{A}) = \underline{I_n}$, then $\underline{A}$ is invertible, and the matrix $\underline{B}$ above is equal to $\underline{A}^{-1}$
   ii) If $(\text{REF } \underline{A}) \neq \underline{I_n}$ (ex: it has a row of all 0's), then $\underline{A}$ is simply noninvertible (and $\underline{B}$ is basically junk)

- Thus, this method both determines if $\underline{A}$ invertible or not, and tells you $\underline{A}^{-1}$ if it exists.
- OTOH, using the determinant $\det(\underline{A})$ (which we'll learn soon) only tells you if $\underline{A}$ is invertible or not:

$$\det(\underline{A}) \neq 0 \implies \underline{A} \text{ invertible}$$
$$\det(\underline{A}) = 0 \implies \underline{A} \text{ noninvertible}$$

**Example**: Determine if $\underline{A}^{-1}$ exists and say what $\underline{A}^{-1}$ is (in the case $\underline{A}$ invertible):

$$\underline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 5 \end{bmatrix}$$

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$\underline{A}$ is $3\times 3$, so augment with $\underline{I_3}$:

$$[\underline{A} \mid \underline{I_3}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 2 & 4 & 5 & 0 & 0 & 1 \end{array}\right]$$

Row reduce until $\underline{A}$ is in REF:

$$\begin{pmatrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_2 \end{pmatrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right]$$

notice this augmented piece essentially keeps a tally of the row ops you've done

$$(R_3 \to R_3 - 4R_2) \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 5 & 10 & -4 & 1 \end{array}\right]$$

$$(R_3 \to \tfrac{1}{5} R_3) \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 2 & -\frac{4}{5} & \frac{1}{5} \end{array}\right]$$

The left-hand side is in REF, and is equal to $\underline{I_3}$.

Thus, $\underline{A}$ is invertible, and

$$\boxed{\underline{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -\frac{4}{5} & \frac{1}{5} \end{bmatrix}}$$

$\underline{Ex}$ Do the same with $\underline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

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$$[\underline{A} \mid \underline{I_3}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$(R_2 \leftrightarrow R_3) \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & \boxed{2} & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -3 & 1 & 0 \end{array}\right]$$

our last move is $R_2 \to \frac{1}{2}R_1$, but we can already see we're not getting $\underline{I_3}$ on the left-hand side since the last row is all 0's. Thus, $\underline{A}$ is <u>noninvertible</u>

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Why does $[\underline{A} \mid \underline{I}] \to [\underline{I} \mid \underline{A}^{-1}]$ work ?

<u>Idea</u>: • Pick $\underline{B}$ so that $\underline{BA} = (\text{REF } \underline{A})$

("what are row operations"?)

• Row ops $\longleftrightarrow$ multiplying by $\underline{B}$.

• $\underline{B}[\underline{A} \mid \underline{I}] = [\underline{BA} \mid \underline{BI}]$

$= [\underline{I} \mid \underline{B}]$

$= [\underline{I} \mid \underline{A}^{-1}]$

## More on inverses

- $(\underline{A}^{-1})^{-1} = \underline{A}$

- Inverses are "unique": there cannot be two different matrices that are both $\underline{A}^{-1}$

$\boxed{\cdot \ (\underline{AB})^{-1} = \underline{B^{-1}A^{-1}}}$ NOT $\underline{A^{-1}B^{-1}}$

Matrix powers : $\underline{A}^k = \underbrace{\underline{A}\,\underline{A} \cdots \underline{A}}_{k \text{ times}}$