Sec 3.4(2) and 3.5 Matrix Inverses

Matrix Algebra: $\underline{A}, \underline{B}, \underline{C}$ (n×n)

1) $\underline{A}+\underline{B} = \underline{B}+\underline{A}$, 2) $\underline{A}(\underline{B}+\underline{C}) = \underline{A}\underline{B}+\underline{A}\underline{C}$

3) $(\underline{A}+\underline{B})\underline{C} = \underline{A}\underline{C}+\underline{B}\underline{C}$, 4) $\underline{A}(\underline{B}\underline{C}) = (\underline{A}\underline{B})\underline{C}$

* 5) $\underline{A}\underline{B} \neq \underline{B}\underline{A}$ generally, even if both products are defined

* 6) k scalar, $\underline{x}$ vector; $\underline{A}(k\underline{x}) = k(\underline{A}\underline{x}) = (k\underline{A})\underline{x}$

$\underline{A}(k\underline{B}) = k(\underline{A}\underline{B}) \ldots$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{\underline{A}}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\underline{I}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$\underline{A}\underline{I} = \underline{A}$

$$\underline{A}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\underline{A}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\underline{I}}\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{\underline{A}} : \begin{matrix} (\text{row } 1) \longrightarrow \\ (\text{row } 2) \longrightarrow \end{matrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \underline{A}$$

so $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\underline{A} = \underline{A}$

Identity Matrices: "1's on the main diagonal"

$\underline{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\underline{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\underline{I_4} = \cdots$

Often we can write "$\underline{I}$" if the size is ~~clear~~ clear.

For all matrices $\underline{A}$, $\boxed{\underline{A}\underline{I} = \underline{I}\underline{A} = \underline{A}}$ $(\underline{I}\underline{x} = \underline{x})$

(assuming products defined)

$3x = 1$
$x \cdot 3 = 1 \iff x = \text{"}3^{-1}\text{"}$

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}} = \underset{AB = I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\underset{B}{\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} = \underset{BA = I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Def: Since $AB = BA = I$, we say A "invertible" (or "nonsingular")
and $B = A^{-1}$ ("A inverse")

- Only square matrices (2×2, 3×3, 4×4, ...) can have inverses.
- Not every A (even squares) will have A^{-1} defined.
- for example, $\emptyset$-matrices $\underset{\emptyset_2}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$, $\underset{\emptyset_3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$, ... won't have inverses.
- A^{-1} undefined $\Longrightarrow$ A is "noninvertible" or "singular"

Special 2×2 formula: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Defin: determinant $\det(A) \overset{\text{DEF}}{=} |A| \overset{\text{DEF}}{=} ad - bc$

★ $\boxed{A \text{ invertible} \iff \det(A) \neq 0}$ invertible $\iff |A| \neq 0$

- If $|A| \neq 0$, then $A^{-1} = \dfrac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\underline{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Is $\underline{A}$ invertible? (Is $\underline{A}^{-1}$ defined?)

$|\underline{A}| = (1)(4) - (2)(3) = -2 \neq 0$ ✓

$$\underline{A}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

- Read "How to find inverse for 3×3"
- Read "Solving systems with inverses"