Solving systems with "inverse method"

Solve the system using inverses $\rightarrow \begin{cases} x_1 + 3x_2 = 9 \\ 2x_1 + x_2 = 8 \end{cases}$

System is same as "vector equation"

$$x_1\begin{bmatrix}1\\2\end{bmatrix} + x_2\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}9\\8\end{bmatrix}$$

$$\begin{bmatrix}1 & 3\\2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}9\\8\end{bmatrix}$$

$$\boxed{\underline{A} \quad \underline{x} \quad = \quad \underline{b}}$$

Solving system for x_1, x_2 is same as solving vector equation $\underline{Ax} = \underline{b}$ for $\underline{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$.

Idea: • IF $\underline{A}^{-1}$ defined, then

$$\underline{Ax} = \underline{b} \rightarrow \underline{A^{-1}Ax} = \underline{A^{-1}b}$$

$$\rightarrow \underline{Ix} = \underline{A^{-1}b} \rightarrow \boxed{\underline{x} = \underline{A^{-1}b}}$$

$\det(\underline{A}) = |\underline{A}| = 1\cdot 1 - 3\cdot 2 = -5 \neq 0$; (so $\underline{A}$ invertible)

$$\underline{A}^{-1} = \frac{1}{-5}\begin{bmatrix}1 & -3\\-2 & 1\end{bmatrix} = \begin{bmatrix}-1/5 & 3/5\\2/5 & 1/5\end{bmatrix}$$

$$\underline{x} = \underline{A^{-1}b}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \frac{-1}{5}\begin{bmatrix}1 & -3\\-2 & 1\end{bmatrix}\begin{bmatrix}9\\8\end{bmatrix} = \frac{-1}{5}\left(\begin{bmatrix}9\\-18\end{bmatrix} + \begin{bmatrix}-24\\8\end{bmatrix}\right)$$

$$= \frac{-1}{5}\begin{bmatrix}-15\\-10\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

Ex: Use inverses to solve system

$$\begin{cases} x_1 & = 3 \\ 3x_1 + x_2 & = 7 \\ 2x_1 + 4x_2 + 5x_3 & = -1 \end{cases}$$

(yes, we can obviously use $x_1 = 3$ and substitute; this example was chosen for simplicity)

the above system is equivalent to the vector equation $\underline{A}\underline{x} = \underline{b}$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 5 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}}_{\underline{b}}$$

we already saw $\underline{A}$ invertible, $\underline{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -\frac{4}{5} & \frac{1}{5} \end{bmatrix}$.

Then

$$\underline{A}\underline{x} = \underline{b} \iff \underline{A^{-1}A\underline{x}} = \underline{A^{-1}b} \iff \underline{x} = \underline{A^{-1}b}$$

$$\underline{x} = \underline{A^{-1}b} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -\frac{4}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 6 - \frac{28}{5} - \frac{1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -2 \\ \frac{1}{5} \end{bmatrix}$$

Notice if I change $\underline{b}$ we can find the new solution fast:

solve $$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Again

$$\underline{x} = \underline{A^{-1}b} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -\frac{4}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -12/5 \end{bmatrix}$$