Sec 3.6 (1) Determinants

$\star \begin{vmatrix} a & b \\ c & d \end{vmatrix} =$

Last time: if $\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\underline{A}) = |\underline{A}| = ad - bc$.

Also, $\boxed{\underline{A}^{-1} \text{ defined} \iff \det(\underline{A}) \neq 0 \quad \left(\begin{matrix}\text{this is true for} \\ 3\times3,\ 4\times4,\ \dots\end{matrix}\right)}$

So now we need a method/formula for 3×3, 4×4, ... determinant

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$\underline{A}x = \underline{b}$

$x = \underline{A^{-1}b}$

Method: "Cofactor Expansion": Pick a row or column to "expand along"

$$\text{Ex:} \quad \underline{A} = \begin{array}{c} \begin{array}{ccc} a_{11} & a_{12} & a_{13} \end{array} \\ \begin{bmatrix} 2 & 0 & 4 \\ 3 & 4 & 2 \\ 0 & 4 & -2 \end{bmatrix} \end{array}$$

compute $\det(\underline{A})$ "along row 1"

det of matrix from deleted (row 1, col 3) from $\underline{A}$

$$\det(\underline{A}) = \underbrace{(2)}_{a_{11}}\underbrace{(+)}_{(-1)^{1+1}}\begin{vmatrix} 4 & 2 \\ 4 & -2 \end{vmatrix} + \underbrace{(0)}_{a_{12}}\underbrace{(-)}_{(-1)^{1+2}}\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} + \underbrace{(4)}_{a_{13}}(+)\begin{vmatrix} 3 & 4 \\ 0 & 4 \end{vmatrix}$$

$$= 2(-8-8) - 0(-6-0) + 4(12-0)$$

$$= -32 + 48 = 16$$

Now compute "along column 1"

$$\begin{array}{c} a_{11} \\ a_{21} \\ a_{31} \end{array}\begin{bmatrix} 2 & 0 & 4 \\ 3 & 4 & 2 \\ 0 & 4 & -2 \end{bmatrix}$$

$$\det(\underline{A}) = (2)(-1)^{1+1}\begin{vmatrix} 4 & 2 \\ 4 & -2 \end{vmatrix} + (3)(-1)^{2+1}\begin{vmatrix} 0 & 4 \\ 4 & -2 \end{vmatrix} + (0)(-1)^{3+1}\underbrace{\begin{vmatrix} 0 & 4 \\ 4 & 2 \end{vmatrix}}_{\text{"the (3,1)-Minor" } M_{31}}$$

$$= 2(-8-8) - 3(0-16) + 0(0-16)$$

$$= -32 + 48 = 16$$

"the (3,1)-Minor" M_{31}

"the (3,1)-cofactor" C_{31}

Expansion along row 1

$a_{11},\ a_{12},\ a_{13}$

$$\det(\underline{A}) = a_{11}\boxed{C_{1,1}} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}\boxed{(-1)^{1+1}M_{11}} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13}$$

Along column 1 :

a_{11}
a_{21}
a_{31}

$$\det(\underline{A}) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= a_{11}(-1)^{1+1}M_{11} + a_{21}(-1)^{2+1}M_{21} + a_{31}(-1)^{3+1}M_{31}$$

- Any choice of row/column yields same value for $\det(\underline{A})$. (so pick one with more 0's)
- The signs included in C_{ij} is $(-1)^{i+j}$, and these have pattern $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Ex: $\underline{A} = \begin{bmatrix} 8 & 0 & 0 & 5 \\ 5 & 8 & 3 & -7 \\ 2 & 0 & 0 & 0 \\ 7 & 2 & 1 & 7 \end{bmatrix}$ (4×4) Compute $\det(\underline{A})$ "along row 3"

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$\det(\underline{A}) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34}$$

$$= (2)(+)M_{31} + \cancel{(0)(-)M_{32}}^{\,0} + \cancel{(0)(+)M_{33}}^{\,0} + \cancel{(0)(-)M_{34}}^{\,0}$$

$$= 2\begin{vmatrix} 0 & 0 & 5 \\ 8 & 3 & -7 \\ 2 & 1 & 7 \end{vmatrix} = 2\left(0 - 0 + 5\begin{vmatrix} 8 & 3 \\ 2 & 1 \end{vmatrix}\right)$$

$$= 2\cdot 5\cdot (8-6) = 20$$

* • Read note "Solving systems w/ Cramer's Rule"

Transpose: Swaps rows $\longleftrightarrow$ columns

$$\underline{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow \underline{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\underline{B} = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{2\times 3} \longrightarrow \underline{B}^T = \underbrace{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}}_{3\times 2}$$

- $(\underline{A}+\underline{B})^T = \underline{A}^T + \underline{B}^T$
- $(\underline{A}^T)^T = \underline{A}$

☆ $(\underline{AB})^T = \underline{B}^T\underline{A}^T$ NOT $\underline{A}^T\underline{B}^T$