More on determinants

Recall Cofactor expansion (3×3)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Pick <u>any</u> row/column

- Row 2 : $|\underline{A}| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

$$= a_{21}(-1)^{2+1}M_{21} + a_{22}(-1)^{2+2}M_{22} + a_{23}(-1)^{2+3}M_{23}$$

Properties : $|\underline{A}|$ can be negative! (orientation flip)

- $|\underline{A}^T| = |\underline{A}|$ (transpose : rows $\longleftrightarrow$ cols)
- $|\underline{A}| \neq 0 \iff \underline{A}^{-1}$ defined
- If $|\underline{A}| \neq 0$, then $|\underline{A}^{-1}| = \dfrac{1}{|\underline{A}|}$.

<u>(upper) triangular matrix</u>

(entries below main diagonal $= 0$)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

<u>(lower) triangular</u>

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

★ det([triangular matrix])

= (product of diagonal entries)

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 1 \cdot 4 \cdot 6 = 24$

Now, can we use row/col ops to make 0's in matrix or make triangular to get $|\underline{A}|$?

Swap : $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{(R_1 \leftrightarrow R_2)} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$\det = -2$ $\qquad\qquad$ $\det = +2 = -(-2)$

(b/c Cofactor signs $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$)

$$\boxed{\text{swapping two rows (or two columns) flips sign of } \det(\cdots)}$$

Note column ops work similarly b/c :

transpose swaps rows $\longleftrightarrow$ columns

$\underline{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{T} \underline{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$(\det = -2)$ $\qquad\qquad$ $(\det = -2)$

$(R_1 \leftrightarrow R_2)$ $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ $\qquad$ $(C_1 \leftrightarrow C_2)$ $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

$(\det = 2)$ $\qquad\qquad$ $(\det = 2)$

Scale : $\boxed{\text{multiplying one row/column by } k \text{ causes det. to get multiplied by } k.}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $(\det = -2)$

$(R_2 \to -3R_2)$ $\begin{bmatrix} 1 & 2 \\ -9 & -12 \end{bmatrix}$ $(\det = (-3)(-2) = 6)$

$\boxed{\begin{array}{l}\underline{\text{adding}}: \text{ Ops } (R_i \to R_i + kR_j) \text{ and } (C_i \to C_i + kC_j) \\ \qquad \text{do } \underline{\text{NOT}} \text{ affect determinant!}\end{array}}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad (\det = -2)$$

$$(R_2 \to R_2 - 3R_1) \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \quad \text{triangular} \Rightarrow (\det = 1 \cdot (-2) = -2)$$

$\underline{Ex}$

$$\underline{A} = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -3 & 2 \\ 5 & 1 & 9 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{pmatrix} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 + 3R_3 \end{pmatrix} \begin{bmatrix} 12 & 0 & 21 \\ 18 & 0 & 29 \\ 5 & 1 & 9 \end{bmatrix} \quad \text{no effect on determinant}$$

column 2

$$\det = -0(\cdots) + 0(\cdots) - 1 \begin{vmatrix} 12 & 21 \\ 18 & 29 \end{vmatrix}$$

$$-1\left((360-12) - (360+18)\right)$$

$$-1(-30) = \boxed{30}$$

$\underline{Ex}$

$$\underline{A} = \begin{bmatrix} 2 & 4 & -2 & 6 \\ 1 & 2 & 5 & 4 \\ 1 & 1 & 2 & 4 \\ 0 & 2 & -6 & 3 \end{bmatrix} \quad \text{nicer \#'s.}$$

$\underline{\text{NET effect}}$

$$(R_1 \leftrightarrow R_3) \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 5 & 4 \\ 2 & 4 & -2 & 6 \\ 0 & 2 & -6 & 3 \end{bmatrix} \qquad \times(-1)$$

$$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{pmatrix} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & -6 & -2 \\ 0 & 2 & -6 & 3 \end{bmatrix} \qquad \times(-1)$$

$$\begin{pmatrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 2R_4 \end{pmatrix} \quad \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -12 & -2 \\ 0 & 0 & -12 & 3 \end{bmatrix} \quad \times(-1)$$

$$(R_4 \to R_4 - R_3) \quad \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -12 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \times(-1)$$

triangular $\Rightarrow$ $\det = -60$. Then $\det(\underline{A}) = (-60)(-1)$

$= \boxed{60}$

<u>Adjugate matrix</u> (for 3×3 — works for $4\times 4, \dots$)

$$\operatorname{adj}(\underline{A}) \overset{\text{DEF}}{=} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$\leftarrow$ cofactors

<u>Inverse matrix thm</u>: If $\det(\underline{A}) \neq 0$, then

$$\boxed{\underline{A}^{-1} = \frac{\operatorname{adj}(\underline{A})}{\det(\underline{A})} = \frac{1}{|\underline{A}|} \cdot \operatorname{adj}(\underline{A})}$$

Why? Ex: $(1,1)$ entry of $\underline{A}\,\operatorname{adj}(\underline{A})$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} C_{11} & \cdots \\ C_{12} & \cdots \\ C_{13} & \cdots \end{bmatrix}$$

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = |\underline{A}|$$

so $\underline{A} * \dfrac{\operatorname{adj}(\underline{A})}{|\underline{A}|}$ makes 1's and 0's $\longrightarrow$ $\underline{I_n}$

Two example test questions to demonstrate the concepts:

Suppose that

$$\underbrace{\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & \boldsymbol{i} \end{bmatrix}}_{A} = 5.$$

Find

$$\underbrace{\det \begin{bmatrix} 2g+3a & 2h+3b & 2\boldsymbol{i}+3c \\ d & e & f \\ 2a & 2b & 2c \end{bmatrix}}_{B}.$$

For this, we want to "trace" row/column ops that take us from $\underline{B}$ back to $\underline{A}$, and record the effect (net) on det.

<u>Ops</u>		<u>net change</u>
	$\begin{vmatrix} 2g+3a & 2h+3b & 2i+3c \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix}$	—
$(R_3 \to \frac{1}{2}R_3)$ $\times 1/2$	$\begin{vmatrix} 2g+3a & 2h+3b & 2i+3c \\ d & e & f \\ a & b & c \end{vmatrix}$	$\times (\frac{1}{2})$
$(R_1 \to R_1 - 3R_3)$ (no effect)	$\begin{vmatrix} 2g & 2h & 2i \\ d & e & f \\ a & b & c \end{vmatrix}$	$\times (1/2)$

$(R_1 \to \tfrac{1}{2} R_1)$ $\times \tfrac{1}{2}$ $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$ $\times (1/2)(1/2)$

$(R_1 \longleftrightarrow R_3)$ $\times(-1)$ $\underline{A} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ $\underline{\times (-1)(1/2)(1/2)}$ $= \times(-1/4)$

$\underline{A}$ has determinant 5, and getting from $\underline{B} \to \underline{A}$ via row/col ops accumulated a total effect of $\times(-1/4)$.

This means that

$$\det(\underline{B}) \cdot \begin{pmatrix}\text{net} \\ \text{effects}\end{pmatrix} = \det(\underline{A}) = 5$$

$$\Longrightarrow \det(\underline{B}) \times (-1/4) = 5$$

$$\Longrightarrow \boxed{\det(\underline{B}) = -20}$$

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1. The matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has $\det A = 5$. Find the $(1,2)$ entry of A^{-1} (the entry in row 1 and column 2 of the matrix A^{-1}).

A. $-\frac{2}{5}$

B. $\frac{2}{5}$

C. 0

D. $\frac{1}{5}$

E. 2

Although we could compute $\underline{A}^{-1}$ "in full" using the $[\underline{A} \mid \underline{I}] \longrightarrow [(\text{REF } A) \mid \underline{A}^{-1}?]$ method, we can get the desired entry using the "Inverse matrix theorem", i.e.

$$\underline{A}^{-1} = \frac{1}{|\underline{A}|} \cdot \operatorname{adj}(\underline{A}) = \frac{1}{|\underline{A}|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \frac{1}{|\underline{A}|} \cdot \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

(note rows/cols flipped)

So the (1,2) entry of $\underline{A}^{-1}$ is just

$$\frac{1}{|\underline{A}|} \cdot C_{21}$$

so we need the (2,1) cofactor $C_{21} = (-1)^{2+1} M_{21}$

$$= (-1) \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}$$ (deleted row 2, col 1)

$$= (-1)(-2) = 2.$$

Then

(1,2) entry of $\underline{A}^{-1}$ is $\frac{1}{|\underline{A}|} C_{21} = \frac{1}{5} \cdot -2$

$$\boxed{= -2/5}$$