For solving system/vector eq we have a few options:

$$\underset{n\times n}{\underline{A}}\,\underset{n\times 1}{\underline{x}} = \underset{n\times 1}{\underline{b}},$$

1) Row reduce augmented matrix $[\underline{A} \mid \underline{b}]$ to (EF) or (REF), look at pivot/free variables, ...
   - works always (no/1/∞-ly many solutions) but doesn't "tweak" well: changing $\underline{b}$ requires you to resolve.
2) If $\underline{A}^{-1}$ is defined, then $\underline{A}\underline{x} = \underline{b}$ has a <u>unique</u> solution $\underline{x} = \underline{A^{-1}b}$
   - Easy to modify $\underline{b}$, but requires computing $\underline{A}^{-1}$ first (assuming it exists!)
3) (New) Cramer's Rule
   Use a determinant method to individually solve for variables $x_1, x_2, \ldots$
   - Faster than it sounds, but requires $|\underline{A}| \neq 0$

Demonstrating Cramer's rule by 2x2 example

Solve system $\longleftrightarrow$ find $\underline{x}$

$$\begin{cases} x_1 + 3x_2 = 9 \\ 2x_1 + x_2 = 8 \end{cases} \longleftrightarrow \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 9 \\ 8 \end{bmatrix}}_{\underline{b}}$$

- First, compute $|\underline{A}| = 1 - 6 = -5$.
- Let $\underline{c}_1, \underline{c}_2$ be the columns of $\underline{A}$ $\left(\underline{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

  so $\underline{A} = [\underline{c}_1 \ \underline{c}_2]$ $\left(\underline{c}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$
- Now build two new matrices

$$\underline{B}_1 = [\underline{b} \ \underline{c}_2] = \begin{bmatrix} 9 & 3 \\ 8 & 1 \end{bmatrix}$$

$$\underline{B}_2 = [\underline{c}_1 \ \underline{b}] = \begin{bmatrix} 1 & 9 \\ 2 & 8 \end{bmatrix}$$

- Cramer's rule gives us a formula for the solutions

$$x_1 = \frac{|\underline{B}_1|}{|\underline{A}|} = \frac{\begin{vmatrix} 9 & 3 \\ 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-15}{-5} = 3$$

$$x_2 = \frac{|\underline{B}_2|}{|\underline{A}|} = \frac{\begin{vmatrix} 1 & 9 \\ 2 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-10}{-5} = 2.$$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is the unique solution for the system.

Works for $3\times3, \ldots$ too.

Ex: Solve system using Cramer's Rule

$$\begin{cases} x_1 + x_2 = 4 \\ -4x_1 + 3x_3 = 0 \\ x_2 - 3x_3 = 3 \end{cases}$$

This system is equivalent to the vector eq

$$\underline{A}\underline{x} = \underline{b} : \begin{bmatrix} 1 & 1 & 0 \\ -4 & 0 & 3 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$

$$\underline{A} \qquad \underline{x} \qquad \underline{b}$$

$|\underline{A}| =$ (along row 1)

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= (1)(+)\begin{vmatrix} 0 & 3 \\ 1 & -3 \end{vmatrix} + (1)(-1)\begin{vmatrix} -4 & 3 \\ 0 & -3 \end{vmatrix} + 0(+)M_{13}$$

$$= (-3) - (12) + 0 = \boxed{-15}$$

- Now write $\underline{A} = [\underline{c_1} \ \underline{c_2} \ \underline{c_3}]$

$$\underline{B_1} = [\underline{b} \ \underline{c_2} \ \underline{c_3}]$$

$$\underline{B_2} = [\underline{c_1} \ \underline{b} \ \underline{c_3}]$$

$$\underline{B_3} = [\underline{c_1} \ \underline{c_2} \ \underline{b}]$$

- Cramer's rule: Solution (x_1, x_2, x_3) has

$$x_1 = \frac{|\underline{B_1}|}{|\underline{A}|}, \quad x_2 = \frac{|\underline{B_2}|}{|\underline{A}|}, \quad x_3 = \frac{|\underline{B_3}|}{|\underline{A}|}$$

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{(-15)} \begin{vmatrix} 4 & 1 & 0 \\ 0 & 0 & 3 \\ 3 & 1 & -3 \end{vmatrix} = \frac{1}{-15}\left(-3 \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix}\right)$$

$$= \frac{1}{5}$$

$$x_2 = \frac{|B_2|}{|A|} = \frac{-1}{15} \cdot \begin{vmatrix} 1 & 4 & 0 \\ -4 & 0 & 3 \\ 0 & 3 & -3 \end{vmatrix} =$$

$$= \frac{-1}{15}\left(1 \begin{vmatrix} 0 & 3 \\ 3 & -3 \end{vmatrix} - 4 \begin{vmatrix} -4 & 3 \\ 0 & -3 \end{vmatrix} + 0\right)$$

$$= \frac{-1}{15}\left(-9 - 48\right)$$

$$= \frac{57}{15} = \frac{19}{5}$$

$$x_3 = \frac{|B_3|}{|A|} = \frac{1}{-15} \begin{vmatrix} 1 & 1 & 4 \\ -4 & 0 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= \frac{-1}{15}\left((-1)(-4) \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}\right)$$

$$= \frac{-1}{15}\left(4(-1)\right) = \frac{4}{15}$$