Sec 4.1 Vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$

Watch 3Blue1Brown video (linked on course page)

Set notation : $\{ (\text{generic element}) : (\text{specific conditions}) \}$

"such that"

"pool/universe"

"in"

$\{ x \in \mathbb{R} : x > 0 \} = (0, \infty)$

$\{ (x,y) \in \mathbb{R}^2 : x > 0, y < 0 \} =$ "4th quadrant"

$\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x \in \mathbb{R}, y = 0 \} =$ "x-axis"

essentially means "x unrestricted"

alt: $\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \}$

- Sets discard duplicates

- Linear combination : a weighted sum of vectors :

  Ex: $2\underline{u} - 3\underline{v}$, $\frac{1}{3}\underline{u} + 5\underline{v} - \underline{w}$

- Span (of some vectors): set of all possible linear comb. of those vectors (i.e, everything that can be "made" as a lin. comb. of them)

$$\text{span}\{\underline{u}, \underline{v}\} = \{ a\underline{u} + b\underline{v} : a, b \in \mathbb{R} \}$$

$\text{span}\{\hat{i}, \hat{j}\} = \mathbb{R}^2$

$\text{span}\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \} = \{ a \begin{bmatrix} 1 \\ 1 \end{bmatrix} : a \in \mathbb{R} \} =$ (line $y = x$)

$\text{span}\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \} = \{ a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \} = \mathbb{R}^2$

$\text{span}\left\{\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\1\\-2\end{bmatrix}\right\} = \left\{a\begin{bmatrix}1\\0\\3\end{bmatrix} + b\begin{bmatrix}0\\1\\-2\end{bmatrix} : a, b \in \mathbb{R}\right\}$

$= \left\{\begin{bmatrix}a\\b\\3a-2b\end{bmatrix} : a, b \in \mathbb{R}\right\}$

$= \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : x, y \in \mathbb{R}, \underbrace{z = 3x - 2y}_{3x-2y-z=0}\right\}$

<u>Def</u>: If S is a set and $\text{span}\{\underline{u}, \underline{v}\} = S$, then "$\{\underline{u}, \underline{v}\}$ spans S", "$\{\underline{u}, \underline{v}\}$ is a spanning set for S."

$\{\hat{\imath}, \hat{\jmath}\}$ is a spanning set for $\mathbb{R}^2$, but not for $\mathbb{R}^3$

<u>Def</u>: $\underline{u}$ and $\underline{v}$ are <u>linearly dependent</u> if $\underline{u} = c\underline{v}$, or $\underline{v} = c\underline{u}$ (or both). If not, they are <u>lin. independent</u>.

"$\{\underline{u}, \underline{v}\}$ is a <u>lin</u> <u>indep</u>/<u>depen</u> set"

<u>Def</u>. 3 or more vectors are <u>lin</u> <u>depen</u> if some one is a <u>linear</u> <u>combination</u> of the others.* If not, they are <u>lin</u> <u>indep</u>.

* alt: one is in the span of the others.

<u>Ex</u>: $\underline{u} = \begin{bmatrix}-1\\2\end{bmatrix}$, $\underline{v} = \begin{bmatrix}7\\8\end{bmatrix}$, $\underline{w} = \begin{bmatrix}4\\3\end{bmatrix}$,

try to write $a\underline{u} + b\underline{v} = \underline{w}$.

Want $a\begin{bmatrix}-1\\2\end{bmatrix} + b\begin{bmatrix}7\\8\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}$

$(A\underline{x} = \underline{b})$ $\begin{bmatrix}-1 & 7\\2 & 8\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix}$

Solve using REF ... find $\underline{x} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$.

so $\boxed{-\frac{1}{2}\underline{u} + \frac{1}{2}\underline{v} = \underline{w}.}$ $\quad (\underline{w} \in \text{span}\{\underline{u}, \underline{v}\})$

$\underline{v} = \underline{u} + 2\underline{w}, \quad \underline{u} = \underline{v} - 2\underline{w}$

With more vectors, question "which is the lin comb?" is harder to guess, and ambiguous.

So easier to instead write

$-\frac{1}{2}\underline{u} + \frac{1}{2}\underline{v} - \underline{w} = \underline{0}$ or $\boxed{\underline{u} - \underline{v} + 2\underline{w} = \underline{0}}$

Ex: Is $\underline{w} = \begin{bmatrix} -2 \\ 5 \\ -13 \end{bmatrix}$ in $\text{span}\{\underset{\underline{u}}{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}, \underset{\underline{v}}{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}\}$?

Same as asking if there are any solutions to the vector eq.

$a\underline{u} + b\underline{v} = \underline{w} \longleftrightarrow a\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ -13 \end{bmatrix}$

$\longleftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ -13 \end{bmatrix}$.

$\Longrightarrow$ reduce $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 3 & -2 & -13 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & -2 & -7 \end{array}\right]$

$\hookrightarrow \begin{array}{c} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{array}\right] \\ a \quad b \end{array} \longrightarrow 0 = 3 \Longrightarrow$ no solutions.

Since no solutions, $\underline{w} = \begin{bmatrix} -2 \\ 5 \\ -13 \end{bmatrix}$ is not in $\text{span}\{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\}$