Sec 5.3 Homog. Eqs with const. coeffs. (part 1)

Previously (Ex) $\{y'' - 1y' - 6y = 0\}$ homog. $(r^2 - r - 6 = 0)$

const. coeffs.

"try" solution $\boxed{y = e^{rx}}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$

plug in $\longrightarrow$ $r^2 e^{rx} - re^{rx} - 6e^{rx} = 0$

$$\underbrace{(r^2 - r - 6)}_{\text{char. poly.}} e^{rx} = 0$$

roots of <u>characteristic polynomial</u> $r^2 - r - 6$

$$(r-3)(r+2) = 0$$

$r = 3$

$r = -2$ occur once each, so they each have "<u>multiplicity</u> one" each

$(r-3)(r+2) = 0$

$y' - 3y = 0$ $\qquad$ $y' = -2y$

$y' = 3y$ $\qquad$ e^{-2x}

e^{3x}

2nd order $\Rightarrow$ 2 basis solutions

$y_1 \quad y_2$

general solution

$y = c_1 y_1 + c_2 y_2$

$r = 3$ piece $\longrightarrow$ $y_1 = e^{3x}$

$r = -2$ piece $\longrightarrow$ $y_2 = e^{-2x}$

general solution $y = c_1 e^{3x} + c_2 e^{-2x}$

<u>Ex</u>: $y'' - 6y' + 9y = 0$

char. eq: $r^2 - 6r + 9 = 0$

$$(r-3)^2 = 0$$

roots are $r = 3, 3$ (multiplicity 2)

$y_1 = e^{3x}$ $\qquad$ $y_2 = xe^{3x}$ basis solutions

general sol. $y = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 x e^{3x}$

Patterns continue for higher order linear homog. ODE w/ const. coeffs.

<u>homog</u>: $y'' - 6y' + 9y = 0$ $\begin{pmatrix} y \to cy \\ y' \to cy' \\ y'' \to cy'' \end{pmatrix} \Longrightarrow \cancel{c}y'' - 6\cancel{c}y' + 9\cancel{c}y = 0$

<u>Ex</u>: $\{ y^{(5)} - 8y^{(4)} + 16y^{(3)} = 0 \}$ homog., $\underbrace{\text{const. coeffs.}}_{\text{(so find roots of char. poly)}}$

char. poly. $r^5 - 8r^4 + 16r^3 = 0$

$r^3(r^2 - 8r + 16) = 0$

$r^3(r-4)^2 = 0$

roots are $r = \underbrace{0, 0, 0}_{\text{multiplicity } 3},\ \overbrace{4, 4}^{\text{multiplicity } 2}$

5^{th} order eq $\Longrightarrow$ need 5 basis solutions, $y_1, y_2, \ldots, y_5$

gen. solution $y = c_1 y_1 + c_2 y_2 + \cdots + c_5 y_5$

Gen solut. $y = [\text{"}r=0 \text{ piece"}] + [\text{"}r=4 \text{ piece"}]$

"$r^3 = 0 \leftrightarrow y''' = 0$" (mult 3) | "$(\cancel{44}\ r-4)^2 = 0$"

$r=0$ piece : $y_1 = e^{0x} = 1$

$y_2 = xe^{0x} = x$

$y_3 = x^2 e^{0x} = x^2$

$r=4$: $y_4 = e^{4x}$

$y_5 = xe^{4x}$

General solution $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 + c_5 y_5$

$y = (c_1 + c_2 x + c_3 x^2) + (c_4 e^{4x} + c_5 x e^{4x})$

$(c_1 + c_2 x + c_3 x^2) + (c_4 + c_5 x)e^{4x}$

Can "work backwards"

Ex: Write a homog. linear ODE with gen solution

$$y = \underbrace{c_1}_{\substack{r=0 \\ (\text{mult. } 1)}} + \underbrace{c_2 e^{-x} + c_3 x e^{-x}}_{\substack{r=-1\ (e^{-x}) \\ (\text{mult. } 2)}} + \underbrace{c_4 e^{5x}}_{\substack{r=5\ (e^{5x}) \\ (\text{mult. } 1)}}$$

char eqn should be $(r-0)(r+1)^2(r-5) = 0$

$$r(r^2+2r+1)(r-5) = 0$$

$$r(r^3+2r^2+r-5r^2-10r-5) = 0$$

$$r^4-3r^3-9r^2-5r = 0$$

ODE $\quad y^{(4)} - 3y^{(3)} - 9y'' - 5y' = 0$

Solving (cubics +) by "inspection/division"

- Sometimes we can find patterns for factoring

$$y^{(3)} + 5y^{(2)} - 100y' - 500y = 0$$

← things without y go here

char eq $\quad \underbrace{r^3 + 5r^2}_{\times 5} - \underbrace{100r - 500}_{\times 5} = 0$

$$r^2(r+5) - 100(r+5) = 0$$

$$(r^2-100)(r+5) = 0$$

$$(r-10)(r+10)(r+5) = 0$$

roots are $r = 10, -10, -5$ (all have multiplicity one)

☆ Read example with polynomial division.

Educated guess + division:

Ex: $3y^{(3)} + 8y'' - 33y' + 10y = 0$

ch. eq: $3r^3 + 8r^2 - 33r + 10 = 0$

Hope for $(r-k)\underbrace{(ar^2 + br + c)}_{\text{can use quad formula.}} = 0$

need this root

Recall Theorem: $a_n r^n + \cdots + a_1 r + a_0 = 0$

Any rational root $r = \frac{p}{q}$ (p, q coprime)

must have $p = \pm(\text{divisor of } a_0)$

$q = \pm(\text{divisor of } a_n)$

(this gives a candidate list)

for $3r^3 + 8r^2 - 33r + 10 = 0$

rational root candidates $\frac{p}{q}$ $p \rightarrow \in \{\pm 1, \pm 2, \pm 5, \pm 10\}$, $q \rightarrow \in \{\pm 1, \pm 3\}$

* always start with $r = \frac{p}{1}$ ($q = 1 \Rightarrow$ integer) (easier)

plug in to ch. eq.

$r = 1 \longrightarrow 3 + 8 - 33 + 10 = -12$ ✗

$r = -1 \longrightarrow -3 + 8 + 33 + 10 = 48$ ✗

$r = 2 \longrightarrow 24 + 32 - 66 + 10 = 0$ ✓

so $r = 2$ is a root

$$\Rightarrow 3r^3 + 8r^2 - 33r + 10 = 0$$
$$= (r-2)(ar^2 + br + c) = 0$$

need to find by polynomial long division

$$
\begin{array}{r|l}
 & 3r^2 + 14r - 5 \\
\hline
r-2 & 3r^3 + 8r^2 - 33r + 10 \\
 & \underline{3r^3 - 6r^2} \\
 & \quad 14r^2 - 33r \\
 & \quad \underline{14r^2 - 28r} \\
 & \qquad -5r + 10 \\
 & \qquad \underline{-5r + 10} \\
 & \qquad\qquad 0
\end{array}
$$

ch. eq. : $(r-2)(3r^2 + 14r - 5) = 0$

$$(r-2)(r+5)(3r-1) = 0$$

$r = 2 \quad r = -5 \quad r = 1/3$