Sec 5.3 (part 2)

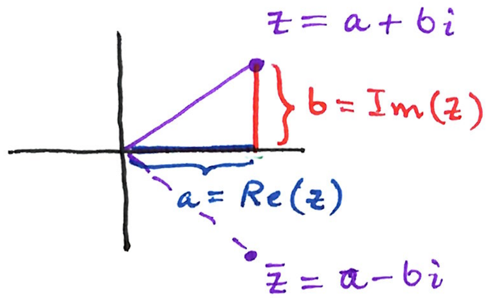
Complex numbers: $z = \underbrace{a}_{\text{Real part}} + \overbrace{b}^{\text{Imaginary Part}} i$

Imaginary Part $\text{Im}(z) = b$

$i^2 = -1$

Real part "$\text{Re}(z)$" $= a$

Conjugation



$z = a + bi$

$\bar{z} = a - bi$

$z + \bar{z} = 2a = 2\,\text{Re}(z) \Rightarrow \text{Re}(z) = \dfrac{z+\bar{z}}{2}$

$z - \bar{z} = 2bi = 2i \cdot \text{Im}(z) \Rightarrow \text{Im}(z) = \dfrac{z-\bar{z}}{2i}$ Don't forget!

Complex Exponential

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

try $$e^{(ix)} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \cdots$$

$i = \sqrt{-1}$ $\quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad i^6 = i^2 = -1, \ldots$

so $$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!}$$

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}_{\text{Imag. part}}$$

(Real part) (Taylor series for $\cos(x)$)

Imag. part (Taylor Series for $\sin(x)$)

$$\boxed{\text{Euler's Formula} \qquad e^{ix} = \cos x + i \sin x}$$

$$\text{Re}(e^{ix}) = \cos x, \quad \text{Im}(e^{ix}) = \sin x$$

Similarly, $e^{-ix} = \cos(-x) + i\sin(-x)$

$$\boxed{e^{-ix} = \cos(x) - i\sin(x)}$$

(conjugate of e^{ix}) is $= e^{-ix}$

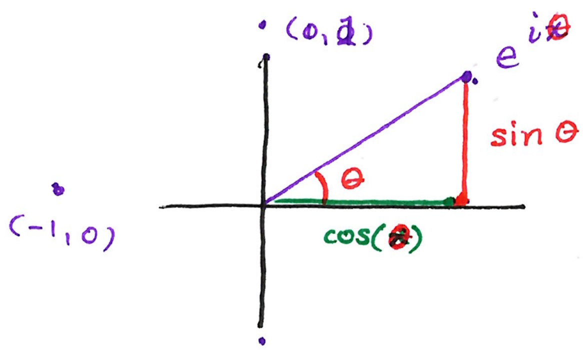
- $\cos(x) = Re(e^{ix}) = \frac{1}{2}(e^{ix}) + \frac{1}{2}(e^{-ix})$
- $\sin(x) = Im(e^{ix}) = \frac{1}{2i}(e^{ix}) - \frac{1}{2i}(e^{-ix})$

$e^{i\pi} = -1 + 0i$

$\cos(\pi) = -1$

$\sin(\pi) = 0$

$e^{i(\pi/2)} = 0 + 1i = i$



$y'' + y = 0$

char. eq. $r^2 + 1 = 0 \Rightarrow r^2 = -1$

roots are $+i, -i$

can also factor $r^2 + 1 = (r-i)(r+i)$

so basis solutions $\left.\begin{array}{l} y_1 = e^{ix} \\ y_2 = e^{-ix} \end{array}\right\}$ lin indep

gen solution $y = c_1 y_1 + c_2 y_2 = \boxed{c_1 e^{ix} + c_2 e^{-ix}}$

Hmm... $y'' + y = 0 \Rightarrow y'' = -y$

$y_1 = \cos x$; $y_1'' = -\cos(x) \Rightarrow y_1 = \cos(x)$ is a solution

$y_2 = \sin x$, $y_2'' = -\sin(x) \Rightarrow y_2 = \sin(x)$ " " " "

so every solution is $y = \tilde{c}_1 \cos(x) + \tilde{c}_2 \sin(x)$

Prefer to use non-complex basis solutions for mechanical problems.

Ex: $y'' + 25y = 0$

char. eq. $r^2 + 25 = 0 \Rightarrow r^2 = -25 \Rightarrow r = \pm 5i$

so gen solution $y = c_1 e^{5ix} + c_2 e^{-5ix}$ $\left(\{e^{ix}, e^{-ix}\}\right)$

"convert" this to $\{\cos x, \sin x\}$ basis solutions

$$y = c_1\left(\cos(5x) + i\sin(5x)\right) + c_2\left(\cos(5x) - i\sin(5x)\right)$$

$$= \underbrace{(c_1 + c_2)}_{\text{"}A\text{"}}\cos(5x) + \underbrace{i(c_1 - c_2)}_{\text{"}B\text{"}}\sin(5x)$$

gen solution can be equivalently expressed as

$$\boxed{y = A\cos(5x) + B\sin(5x)}$$

★ Notice the basis solutions $\{\cos(5x), \sin(5x)\}$ can be obtained by simply using $\text{Re}(e^{5ix})$, $\text{Im}(e^{5ix})$ (let's you skip conversion)

Read the examples in posted.

(incl. repeated complex roots)

Theorem: If $y(x) = a(x) + b(x)i$ (so $y(x)$ is a complex-valued function) and y is a solution of a homogeneous, linear ODE, then $a(x) = \mathrm{Re}(y(x))$ and $b(x) = \mathrm{Im}(y(x))$ are two solutions to the same ODE. (a and b may or may not be linearly indep)

Idea: Example Eq $\{y'' + 5y' - 7y = 0\}$. Because $y = a+bi$ (a, b depending on x)

then
$$y = a + bi$$
$$y' = a' + b'i$$
$$y'' = a'' + b''i$$

Since y is a solution, then

$$y'' + 5y' - 7y = 0$$

$$\Rightarrow (a'' + b''i) + 5(a' + b'i) - 7(a + bi) = 0 + 0i$$

$$\Rightarrow \underbrace{(a'' + 5a' - 7a)}_{=0} + i\underbrace{(b'' + 5b' - 7b)}_{=0} = 0 + 0i$$

(meaning a is a solution for same ODE) (b is a solution for the same ODE)

Then, the specific form $\{\blacksquare'' + 5\blacksquare' - 7\blacksquare = 0\}$ didn't matter; this idea would work with different #'s, or y''', etc.

Use: Gen solution for $\{y'' + 9y = 0\}$. char poly $r^2 + 9 = 0$

$\Rightarrow r = \pm 3i$, so $y_1 = e^{3ix}$, $y_2 = e^{-3ix}$ are two solutions. Alternative,

$$\left.\begin{aligned} \tilde{y}_1 &= \mathrm{Re}(e^{3ix}) = \cos(3x) \\ \tilde{y}_2 &= \mathrm{Im}(e^{3ix}) = \sin(3x) \end{aligned}\right\} \text{ two other solutions, AND lin. indep!}$$

Since eq is 2nd order, 2 lin indep solutions is enough, so

gen solution $y = c_1\tilde{y}_1 + c_2\tilde{y}_2 = \boxed{c_1\cos(3x) + c_2\sin(3x)}$ works

Notes ☆

- complex roots always come in conjugate pairs (Ex: $0 \pm 5i$, $3 \pm 2i$, $\pi \pm 7i$, ...)
- conjugate pairs of roots produce pairs of solutions:

$$\boxed{\text{root pair } r = a \pm bi}$$

$\longrightarrow$ solution pair $c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x}$

$$= e^{ax}\left(\underline{c_1 e^{i(bx)} + c_2 e^{i(-bx)}}\right)$$

convert ...

$$= e^{ax}\left(A\cos(bx) + B\sin(bx)\right)$$

Alt: Use solution $e^{(a+bi)x}$ Real and Imag. parts as basis solutions

$Re(e^{(a+bi)x}) = e^{ax}\cos(bx)$

$Im(e^{(a+bi)x}) = e^{ax}\sin(bx)$

$$= \boxed{Ae^{ax}\cos(bx) + Be^{ax}\sin(bx)}$$

("$a \pm bi$ piece")

Ex: $y^{(3)} - 3y'' + 25y' - 75y = 0$

ch. eqn $r^3 - 3r^2 + 25r - 75 = 0$

$\times 3$ $\times 3$

$$r^2(r-3) + 25(r-3) = 0$$

$$\underbrace{(r-3)}_{r=3}\underbrace{(r^2+25)}_{r=\pm 5i} = 0$$

$r = 3$ mult 1 $\qquad r = \pm 5i$ mult 1 (alt: $r = 0 \pm 5i$)

So gen solution is

$$y = (3 \text{ piece}) + (\pm 5i \text{ piece})$$

$$= c_1 e^{3x} + \left(c_2 \underline{\cos(5x)} + c_3 \underline{\sin(5x)}\right)$$

(we skipped "conversion")

☆ for $\cos(5x)$, $\sin(5x)$ we basically took <u>real</u> and <u>imaginary</u> parts of solution

$$e^{i(5x)} = \boxed{\cos(5x)} + i\boxed{\sin(5x)}$$

<u>Read "order 4 example"</u> **(Next page)**

Order 4 example

Ex: $2y^{(4)} + 7y'' + 3 = 0$

ch. eq: $2r^4 + 7r^2 + 3 = 0$

(only powers of r^2 present)

trick: $s = r^2$

ch. eq $\longrightarrow$ $2s^2 + 7s + 3 = 0$

quad form $\Longrightarrow$ $s = \dfrac{-7 \pm \sqrt{49-24}}{4}$

$$s = \frac{-7 \pm \sqrt{25}}{4} = \frac{-7 \pm 5}{4} \Longrightarrow s = -3 \text{ or } s = -\frac{1}{2}$$

$s = -3 \longrightarrow r^2 = -3 \Rightarrow \boxed{r = \pm\sqrt{3}\, i}$

$s = -1/2 \longrightarrow r^2 = -1/2 \Rightarrow \boxed{r = \pm \frac{1}{\sqrt{2}}\, i}$

} two conjugate pairs

so general solution is

$$y = (\pm\sqrt{3}\, i \text{ piece}) + (\pm \tfrac{1}{\sqrt{2}}\, i \text{ piece})$$

$$= \left(c_1 \underbrace{\boxed{\cos(\sqrt{3}x)}} + c_2 \underbrace{\boxed{\sin(\sqrt{3}x)}}\right)$$

$Re(e^{\sqrt{3}ix}) = \cos(\sqrt{3}x)$

$Im(e^{\sqrt{3}ix}) = \sin(\sqrt{3}x)$

$$+ \left(c_3 \underbrace{\cos\left(\tfrac{1}{\sqrt{2}}x\right)} + c_4 \underbrace{\sin\left(\tfrac{1}{\sqrt{2}}x\right)}\right)$$

$Re(e^{ix/\sqrt{2}}) = \cos(x/\sqrt{2})$

$Im(e^{ix/\sqrt{2}}) = \sin(x/\sqrt{2})$