Sec 5.5 Nonhomog Eqs (part 1)

Recall: $y'' + a(x)y' + b(x)y = f(x)$

assoc. homog. eqn. $y'' + a(x)y' + b(x)y = 0$

Say $y'' - 7y' + 3y = 0$, and $z'' - 7z' + 3z = 0$

$L[\blacksquare] = \blacksquare'' - 7\blacksquare' + 3\blacksquare$ $\quad L[y] = 0 \quad L[z] = 0$

y & z are solutions to the ODE $\{L[\cdot] = 0\}$

Q: Is $(y+z)$ also a solution to the same ODE?

would need $L[y+z] = 0$.

$$(y+z)'' - 7(y+z)' + 3(y+z) \stackrel{?}{=} 0$$

$$y'' + z'' - 7y' - 7z' + 3y + 3z \stackrel{?}{=} 0$$

$$\underbrace{(y'' - 7y' + 3y)}_{L[y]} + \underbrace{(z'' - 7z' + 3z)}_{L[z]} \stackrel{?}{=} 0.$$

$$\underset{0}{L[y]} + \underset{0}{L[z]} \stackrel{?}{=} 0 + 0 = 0$$

So $L[y+z] = 0$, meaning $(y+z)$ is a solution too

Suppose now template L is same $\left(L[\cdot] = (\cdot)'' - 7(\cdot)' + 3(\cdot)\right)$

but ODE is now $L[\cdot] = x$

Suppose $L[y] = x \quad L[z] = x$.

will $L[y+z] \stackrel{?}{=} x$ ... No... $L[y+z] = L[y] + L[z]$

$$= x + x = 2x$$

For ODE $y'' + a(x)y' + b(x)y = f(x)$ (nonhomog.)

a "complete" general solution is

$$y = \boxed{y_c} + \boxed{y_p}$$

general solution to $L[y]=0$ "complementary function"

any particular solution for $L[y]=f$.

Method of Undetermined coeffs (by example)

Problem: Find the (complete) general solution for

$$\underbrace{y'' - 4y}_{L[y]=y''-4y} = 3x^2$$

Step 1 Find general solution y_c for $y''-4y=0$ $(L[y]=0)$,

char poly $r^2-4=0 \Rightarrow r=\pm 2$

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

Step 2 Use trial function y_p that should "filter through $L[\cdot]$" and result/leave desired $f(x)=3x^2$

→ want $L[y_p] = 3x^2$.

- "want y_p to have same 'form' as target $f(x)$"
- $f(x) = 3x^2$ is degree-2 polynomial, let's trial

$$y_p = Ax^2 + Bx + C$$

$y_p = Ax^2 + Bx + C$ Want $y_p'' - 4y_p = 3x^2$

$y_p' = 2Ax + B$

$y_p'' = 2A$

Plug in : $L[y_p] = 2A - 4(Ax^2 + Bx + C) \stackrel{?}{=} 3x^2$

$\underline{-4Ax^2} - \underline{4Bx} + (2A - 4C) = \underline{3x^2}$

$-4A = 3 \Rightarrow A = -3/4$

$-4B = 0 \qquad B = 0$

$2A - 4C = 0 \qquad -6/4 - 4C = 0 \Rightarrow C = -6/16 = -3/8$

so $y_p = -\frac{3}{4}x^2 + 0x - \frac{3}{8}$ is a solution for

$L[y] = 3x^2$ a.k.a $y'' - 4y = 3x^2$.

Step 3 General solution for $y'' - 4y = 3x^2$

is $y = y_c + y_p$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-2x} + \left(-\frac{3}{4}x^2 - \frac{3}{8}\right)}$$

Ex : $\underbrace{y'' - y' - 2y}_{L[y]} = \underbrace{2e^{3x}}_{f(x)}$

Step 1 Compute solution for $L[y] = 0$

$y'' - y' - 2y = 0 \qquad r^2 - r - 2 \quad (r-2)(r+1)$

$\cdots \quad y_c = c_1 e^{-x} + c_2 e^{2x}$

Step 2 Use trial function to "filter through":

$f(x) = 2e^{3x} \qquad y_p = Ae^{3x}$

$$y_p = Ae^{3x}$$
$$y_p' = 3Ae^{3x}$$
$$y_p'' = 9Ae^{3x}$$

$$y_p'' - y_p' - 2y_p \stackrel{?}{=} 2e^{3x}$$
$$9Ae^{3x} - 3Ae^{3x} - 2Ae^{3x} = 2e^{3x}$$
$$4Ae^{3x} = 2e^{3x}$$

$$A = 1/2 \implies y_p = \frac{1}{2}e^{3x}.$$

Step 3 $\quad y = y_c + y_p$

$$\boxed{y = \left(c_1 e^{-x} + c_2 e^{2x}\right) + \left(\frac{1}{2}e^{3x}\right)}$$