Nonhomogeneous Eqs (cont.)

Recap: $\underbrace{y'' - y' - 2y}_{\text{"template" } L[y]} = \cos(x)$ $\qquad (ky)' = ky'$

$x \to y(x) \qquad y \to (L[y])(x)$

★ • $L[y_1 + y_2] = L[y_1] + L[y_2]$

★ • $L[ky] = k \cdot L[y]$

"linear operator"
(any LHS of a linear ODE has these properties)

Solving eq $y'' - y' - 2y = \cos(x)$

$\longleftrightarrow$ find y : $L[y] = \cos(x)$

Step 1 Find general solution y_c for associated homog. eq. $y'' - y' - 2y = 0 \longleftrightarrow L[y_c] = 0$

$\cdots \quad y_c = c_1 e^{-x} + c_2 e^{2x}$

Step 2 Find/use a trial function y_p such that $L[y_p] = \cos(x)$.

★ if $f(x)$ (in $L[y] = f$) has $\cos(x)$ or $\sin(x)$, then include both cos and sin in trial y_p. $f(x) = \cos(x)$

$y_p = A\cos(x) + B\sin(x)$

$y_p' = -A\sin(x) + B\cos(x)$

$y_p'' = -A\cos(x) - B\sin(x)$

$L[y_p] = (-A\cos(x) - B\sin x) - (-A\sin x + B\cos x) - 2A\cos x - 2B\sin x$

$= (-3A - B)\cos x + (A - 3B)\sin x = 1\cos x$

$-3A - B = 1$
$A - 3B = 0$ $\Rightarrow A = \frac{-3}{10},\ B = \frac{-1}{10}$

$y_p = -\frac{3}{10}\cos x + \frac{-1}{10}\sin x$

Step 3 Full general solution for $L[y] = \cos(x)$ aka $y'' - y' - 2y = \cos(x)$ is $y_c + y_p$

$$\boxed{y_c + y_p = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{10}\cos x - \frac{1}{10}\sin x}$$

b/c $L[y_c + y_p] = L[y_c] + L[y_p]$
$= 0 + \cos(x)$

- If RHS $f(x)$ has $\cosh(x)$ or $\sinh(x)$, then use <u>both</u> $\cosh(x)$ and $\sinh(x)$ in trial y_p.
- $\cosh(x)$ "catenary" function 
- Read examples for hyperbolic trig functions in notes.

<u>Ex</u> $\underbrace{y'' - y' - 2y}_{L[y]} = xe^{3x} \longleftrightarrow L[y] = xe^{3x}$

<u>Step 1</u> $\underline{L[y_c] = 0}$ (find general solution for assoc. homog. eq)

$\cdots \quad y_c = c_1 e^{-x} + c_2 e^{2x}$

<u>Step 2</u> Want trial function y_p : $L[y_p] = xe^{3x}$

- Rather than trial $y_p = Axe^{3x}$ $\quad (xe^{3x})' = e^{3x} + xe^{3x}$

use $y_p = Axe^{3x} + Be^{3x} = \underline{(Ax+B)e^{3x}}$

$f(x) = x\boxed{e^{3x}} \longrightarrow$ else e^{3x}

Deg 1 poly $\Rightarrow$ use $(Ax+B)$

<u>Ex</u> $y'' - y' - 2y = x\sin(2x)$ $\quad (x\sin(2x))' \to \sin(2x) + 2x\cos(2x)$

- $y_c = c_1 e^{-x} + c_2 e^{2x}$
- trial? $f(x) = x\sin(2x)$ $\longrightarrow$ use $C\cos(2x) + D\sin(2x)$

Deg 1 poly $\Rightarrow$ include $(Ax+B)$

total trial $y_p = (Ax+B)(C\cos(2x) + D\sin(2x))$

$\boxed{AC}x\cos(2x) + \boxed{BC}\cos(2x) + \boxed{AD}x\sin(2x) + \boxed{BD}\sin(2x)$

$y_p = Ax\cos(2x) + B\cos(2x) + Cx\sin(2x) + D\sin(2x)$

$(3x+2)(\sin(2x))$

$\underline{Ex}$: $\underbrace{y'' - y' - 2y}_{L[y]} = 2e^{3x} + \cos(5x)$

1) want y_c such that $L[y_c] = 0$

2) Find y_p : $L[y_p] = 2e^{3x}$

trial $\longrightarrow$ $y_p = Ae^{3x}$

3) Find $\tilde{y}_p$: $L[\tilde{y}_p] = \cos(5x)$

$$\tilde{y}_p = B\cos(5x) + C\sin(5x)$$

why does this work?

Because $L[y_p + \tilde{y}_p] = L[y_p] + L[\tilde{y}_p] = 2e^{3x} + \cos(5x)$

"full" trial is $Ae^{3x} + B\cos(5x) + C\sin(5x)$