Nonhomog. eqns (cont.)

★ [Lesson 29] Variation of parameters to be read/watched for end of semester.

"Problem case" $\underbrace{y'' - y' - 2y}_{L[y]} = e^{2x}$

- Step 1 $(L[y_c] = 0)$, $y_c = c_1 e^{-x} + c_2 e^{2x}$ $\{e^{-x}, e^{2x}\}$

- Step 2 $(L[y_p] = f = e^{2x})$ $f = e^{2x}$, trial $y_p = Ae^{2x}$

$$L[Ae^{2x}] = A\,L[e^{2x}] = A \cdot 0 = 0.$$

  - Problem is that e^{2x} in trial is a duplicate of the basis solution e^{2x} from y_c.
  - To "remove duplication", we change our trial function to $y_p = Axe^{2x}$ (there is no duplication now)

- Now want $L[Axe^{2x}] = e^{2x}$

$$y_p = Axe^{2x}$$
$$y_p' = Ae^{2x} + 2Axe^{2x}$$
$$y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} = 4Ae^{2x} + 4Axe^{2x}$$

$$L[Axe^{2x}] (= y_p'' - y_p' - 2y_p) = 4Ae^{2x} + 4Axe^{2x} - Ae^{2x} - 2Axe^{2x} - 2Axe^{2x}$$

$$= 3Ae^{2x} \stackrel{?}{=} e^{2x} \to A = 1/3,$$

so $y_p = \frac{1}{3}xe^{2x}$ is a particular solution for $L[\cdot] = e^{2x}$.

Ex: In solving nonhomog. ODE

$$y^{(5)} - 6y^{(4)} + 9y^{(3)} = x + 2e^{3x}.$$

what is the appropriate form of a trial function y_p?

- Step 1 ($L[y_c] = 0$) char. eq. $r^5 - 6r^4 + 9r^3 = 0$

$$r^3(r-3)^2 = 0 \qquad r^3(r^2 - 6r + 9)$$

$r = 0$ (mult. 3) $\longrightarrow$ basis solutions $\{1, x, x^2\}$

$r = 3$ (mult. 2) $\longrightarrow$ " " $\{e^{3x}, xe^{3x}\}$

So general $y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{3x} + c_5 x e^{3x}$

Step 2 ($L[y_p] = f$). $f = x + 2e^{3x}$

- First find y_p for $L[y_p] = x$
  - initial trial function $y_p = \overset{1}{A} + B\underline{x}$ or $(Ax + B)$

    duplication here with basis solutions $\{1, x\}$

    change to $y_p = A\underline{x} + B\underline{x^2}$ still duplication

    $\longrightarrow$ $y_p = Ax^2 + Bx^3$ " "

    $\longrightarrow$ $\underline{y_p = Ax^3 + Bx^4}$ No duplication

- Now need trial for $L[\tilde{y}_p] = 2e^{3x}$

  initial trial $\tilde{y}_p = Ce^{3x}$ duplication with e^{3x}

  $\longrightarrow$ $\tilde{y}_p = Cx^2 e^{3x}$

- Full trial solution/function is

$$y_p + \tilde{y}_p = \boxed{Ax^3 + Bx^4 + Cx^2 e^{3x}}$$

Hyperbolic trig functions

$$\{y''+y=0\} \to r^2=-1$$
$$r=\pm i$$

$$\cosh(x) \overset{\text{DEF}}{=} \frac{e^{x}+e^{-x}}{2} \qquad \cos(x) = \frac{e^{ix}+e^{-ix}}{2}$$

$$\sinh(x) \overset{\text{DEF}}{=} \frac{e^{x}-e^{-x}}{2} \qquad \sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$$

"vice versa" $e^{x} = \cosh(x)+\sinh(x)$ | $e^{ix} = \cos(x)+i\sin(x)$

$e^{-x} = \cosh(x)-\sinh(x)$ $\quad$ $e^{-ix} = \cos(x)-i\sin(x)$

$$\frac{d}{dx}\cosh(x) = +\sinh(x)$$

$$\frac{d}{dx}\sinh(x) = +\cosh(x)$$

so $\{\cosh x, \sinh x\}$ are two indep. basis solutions for ODE $\{y''=y\}$

$y''-y=0 \to r^2-1=0 \quad r=\pm 1$

$y=c_1e^{+x}+c_2e^{-x} \quad \{e^{x}, e^{-x}\}$

- Read example on removing duplication w/ $\cosh x$, $\sinh x$ in the notes.

Removing duplication with cosh, sinh:

Ex: $\underbrace{y'' - 25y}_{L[y]} = \cosh(5x)$

Step 1 ($L[y_c]=0$) char eq $r^2 - 25 = 0 \Rightarrow r = \pm 5$

$y_c = c_1 e^{5x} + c_2 e^{-5x}$. Basis solutions $\{e^{5x}, e^{-5x}\}$

For step 2, finding y_p for $L[y_p] = \cosh(5x)$, we have two options, each with pro's/cons.

Step 2 ($L[y_p] = f$) Option 1: convert cosh/sinh to exponentials

Since $\cosh(5x) = \frac{1}{2}e^{5x} + \frac{1}{2}e^{-5x}$, the eq we are trying to solve is $L[y_p] = \frac{1}{2}e^{5x} + \frac{1}{2}e^{-5x}$.

Then our initial trial function would be

$y_p = Ae^{5x} + Be^{-5x}$, but this has duplication with $\{e^{5x}, e^{-5x}\}$.

so try $\boxed{y_p = Axe^{5x} + Bxe^{-5x}}$

$$y_p' = Ae^{5x} + 5Axe^{5x} + Be^{-5x} - 5Bxe^{-5x}$$
$$= A(1+5x)e^{5x} + B(1-5x)e^{-5x}$$
$$y_p'' = 5Ae^{5x} + 5Ae^{5x} + 25xe^{5x} - 5Be^{-5x} - 5Be^{-5x} + 25Bxe^{-5x}$$
$$L[y_p] \quad = 10Ae^{5x} + 25Axe^{5x} - 10Be^{-5x} + 25Be^{-5x}$$

$$y_p'' - 25y_p = 10Ae^{5x} + 25Axe^{5x} - 10Be^{-5x} + 25Be^{-5x} - 25Axe^{5x} - 25Bxe^{-5x}$$
$$= 10Ae^{5x} - 10Be^{-5x}.$$

Need $L[y_p] = \frac{1}{2}e^{5x} + \frac{1}{2}e^{-5x}$, so $A = \frac{1}{20}$, $B = \frac{-1}{20}$

$y_p = \frac{1}{20}xe^{5x} - \frac{1}{20}xe^{-5x}$. Notice this is $= \frac{1}{10}x\sinh(5x)$.

Step 2 ($L[y_p]=f$) Option 2: Use cosh, sinh

In step 1, $y_c = c_1 e^{5x} + c_2 e^{-5x}$. Since $e^{x} = \cosh x + \sinh x$, $e^{-x} = \cosh x - \sinh x$,

we can convert

$$\begin{aligned} y_c &= c_1(\cosh(5x) + \sinh(5x)) \\ &\quad + c_2(\cosh(5x) - \sinh(5x)) \\ &= (c_1+c_2)\cosh(5x) + (c_1-c_2)\sinh(5x) \\ &\doteq \text{“}c_1\text{”}\cosh(5x) + \text{“}c_2\text{”}\sinh(5x). \end{aligned}$$

Basically we use $\{\cosh(5x), \sinh(5x)\}$ as our basis solutions instead of $\{e^{5x}, e^{-5x}\}$. So our y_c

$$y_c = c_1\cosh(5x) + c_2\sinh(5x).$$

Now for trial y_p, we should include <u>both</u> cosh, sinh:

initial $y_p = A\cosh(5x) + B\sinh(5x)$, but this has duplication w/ y_c.

So try $y_p = Ax\cosh(5x) + Bx\sinh(5x)$

$$\begin{aligned} y_p' &\doteq A\cosh(5x) + 5Ax\sinh(5x) \\ &\quad + B\sinh(5x) + 5Bx\cosh(5x) \end{aligned}$$

$$\begin{aligned} y_p'' &= 5A\sinh(5x) + 5A\sinh(5x) + 25Ax\cosh(5x) \\ &\quad + 5B\cosh(5x) + 5B\cosh(5x) + 25Bx\sinh(5x) \\ &= 10A\sinh(5x) + 10B\cosh(5x) + 25Ax\cosh(5x) \\ &\quad + 25Bx\sinh(5x). \end{aligned}$$

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so $L[y_p] = y_p'' - 25y_p = \cdots = 10A\sinh(5x) + 10B\cosh(5x)$.

want $L[y_p] = \cosh(5x)$, so $A=0$, $B=\frac{1}{10}$, and

$$y_p = \frac{1}{10}x\sinh(5x),$$ just like from Option 1.