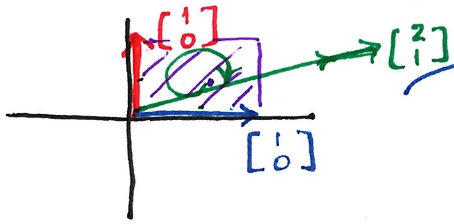
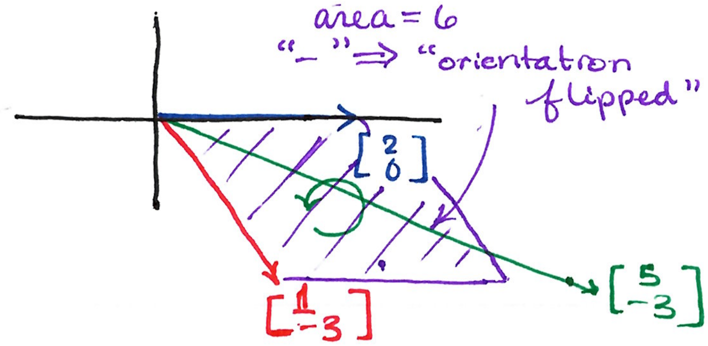
Sec 6.1 Eigenvalues / Eigenvectors

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ det = 1



$\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$ $\det(\cdot) = -6$

area = 6
"−" ⟹ "orientation flipped"



$A = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\underline{e}_1 + \underline{e}_2$

$L[y_1 + y_2] = L[y_1] + L[y_2]$

$\underline{A}(\underline{v}) = \underline{A}(2\underline{e}_1 + \underline{e}_2) = 2(\underline{A}\underline{e}_1) + \underline{A}\underline{e}_2$

$\underline{Ae_1} = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\underline{Ae_2} = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\underline{Av} = 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

Notice that $\underline{Ae_1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\underline{e}_1$

$\underline{Ae_2} = e_1 - 3\underline{e}_2$

<u>Definition</u>: Given $\underline{A}$, a nonzero vector $\underline{v}$ is an <u>eigenvector</u> of $\underline{A}$ with eigenvalue λ if $\boxed{\underline{Av} = \lambda \underline{v}}$

$\lambda \underline{v} = \lambda \underline{I} v$

equivalently if $\underline{Av} - \lambda \underline{I}\underline{v} = \boxed{(\underline{A} - \lambda \underline{I})\underline{v} = \emptyset}$

By example: $\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, what are eigenvalues and eigenvectors?

- We want λ such that there are nonzero solutions for eq $(\underline{A} - \lambda I)\underline{v} = \underline{0}$

$$\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\underline{A}x = \underline{b}$

$\cancel{x = \underline{A}^{-1}\underline{b}}$

- This happens if $\det(\underline{A} - \lambda I) = 0$

$$\det(\underline{A} - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4$$
$$= 4 - 5\lambda + \lambda^2 - 4$$
$$= \lambda^2 - 5\lambda$$

Def: This $\det(\underline{A} - \lambda I)$ is the "characteristic polynomial" (of $\underline{A}$)

The eigenvalues of $\underline{A}$ are the roots of the char. poly.

$\rightarrow \lambda^2 - 5\lambda = \lambda(\lambda - 5) \rightarrow$ roots are $\lambda = 0$ (mult 1)
$\lambda = 5$ (mult. 1)

For $\lambda = 0$, we want $\begin{bmatrix} a \\ b \end{bmatrix}$ to make $(\underline{A} - 0I)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(so $\begin{bmatrix} a \\ b \end{bmatrix} \in \text{null}(\underline{A} - 0I)$)

$$\rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b free

$a = \cancel{1/4}\ -2b$

so solutions are $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2b \\ b \end{bmatrix} = b\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. So since we just want a single eigenvector, we take $b = 1$, and

$$\underline{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

$$\underline{A}v_1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvectors / Eigenvalues (cont.)

<u>Ex</u> $\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, compute/find eigenvalues/vectors for $\underline{A}$

<u>Step 1</u> Find eigenvalues: compute roots λ of char. poly. $\det(\underline{A} - \lambda I)$. $|\underline{A} - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = \cdots = \lambda(\lambda - 5)$

so $\lambda = 0$ (mult 1)
$\lambda = 5$ (" ") are eigenvalues.

<u>Step 2</u> Find an eigenvector for $\lambda = 0$:

Compute null($A - 0I$)

Find $\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ such that $(\underline{A} - 0I)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

... solutions are $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot b$ (b free)

so take $\underline{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, this is an eigenvector (of $\underline{A}$) w/ eigenvalue $\lambda = 0$. $\text{null}(A - 0I) = \left\{ b\begin{bmatrix} -2 \\ 1 \end{bmatrix} : b \in \mathbb{R} \right\}$ → dim = 1.

Compute null($A - 5I$)

Find a eigenvector for $\underline{\lambda = 5}$: want $\begin{bmatrix} a \\ b \end{bmatrix}$ for

$$(\underline{A} - 5I)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow 2a - b = 0 \longrightarrow a = \frac{b}{2}$

... solutions are $\begin{bmatrix} b/2 \\ b \end{bmatrix} = b\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, (b free)

can pick $b = 2 \longrightarrow \underline{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\text{null}(A - 5I) = \left\{ b\begin{bmatrix} 1 \\ 2 \end{bmatrix} : b \in \mathbb{R} \right\}$

- Def: The multiplicity of λ as a root of the char. poly. is the "<u>algebraic multiplicity</u>" of λ.

- <u>Def</u> $E_\lambda = \text{null}(\underline{A} - \lambda I)$ subspace of eigenvectors w/ eigenvalue λ

"geometric multiplicity" of λ is def: $\dim(E_\lambda)$

Ex: $\underline{A} = \begin{bmatrix} 7 & 1 \\ 0 & 7 \end{bmatrix}$ compute e-vals/vects.

Step 1 $\quad |\underline{A}-\lambda I| = \begin{vmatrix} 7-\lambda & 1 \\ 0 & 7-\lambda \end{vmatrix} = (7-\lambda)^2.$

So roots are $\lambda = 7, 7$ (alg. mult. 2)

Step 2 Compute $E_7 = \text{null}(A-7I) = \text{null}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

want $\begin{bmatrix} a \\ b \end{bmatrix}$ for $\begin{bmatrix} 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow b = 0$

a free

solutions are $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (a free)

$E_7 = \left\{ a\begin{bmatrix} 1 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}$ $\quad$ dim $E_7 = 1$.

"geometric mult. is 1"

pick $\underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector

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☆ geom. mult. is always $\leq$ alg. mult.