"Sec 7.4" Repeated eigenvalues / generalized eigenvectors:

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

"generalized" e-vects ("virtual-" / "chain-" / "quasi-")

**eigenvector**: $\underline{A} = \begin{bmatrix} \boxed{2} & 0 \\ 0 & \boxed{7} \end{bmatrix}$ $\quad e_1 \xmapsto{A} 2e_1 \quad e_2 \xmapsto{A} 7e_2 \quad \left( e_1 \xmapsto{\boxed{A-2I}} 0 \quad e_2 \xmapsto{\boxed{A-7I}} 0 \right)$

$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 7-\lambda \end{vmatrix}$

$\underline{A} = \begin{bmatrix} \boxed{3} & 0 & 0 \\ 0 & \boxed{3} & 0 \\ 0 & 0 & \boxed{0} \end{bmatrix}$

$\left.\begin{array}{l} e_1 \xmapsto{A-3I} 0 \\ e_2 \xmapsto{A-3I} 0 \end{array}\right\}$ eigenvectors w/ $\lambda = 3$

$\left. e_3 \xmapsto{A-0I} 0 \right\}$ eigenvector w/ $\lambda = 0$

$Ae_3 = 0e_3 = 0$

$A = \begin{bmatrix} \boxed{3} & 1 \\ 0 & 3 \end{bmatrix}$ $\quad \lambda = 3, 3$ (mult 2) $\quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$E_3 = \left\{ s\begin{bmatrix} 1 \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\}$

"$\underline{\hat{e}_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a "true" eigenvector b/c $\underline{\hat{e}_1} \xmapsto{A-3I} 0$"

$\underline{Ae_2} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \underline{e_1} + 3\underline{e_2}$

$(A-3I)e_2 = e_1 \qquad e_2 \xmapsto{A-3I} e_1$ ($\neq 0$ ~~b/c~~ so not an eigenvect)

but $\quad e_2 \xmapsto{A-3I} e_1 \xmapsto{A-3I} 0 \quad$ "2-step eigenvector"

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$\begin{bmatrix} \boxed{5} & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

$e_1 \xmapsto{A-5I} 0$ (eigenvector)

$e_2 \xmapsto{A-5I} e_1 \xmapsto{A-5I} 0$ (2 step eigenvector)

$e_3 \xmapsto{A-5I} e_2 \xmapsto{A-5I} e_1 \xmapsto{A-5I} 0$ (3 step eigenvector)

$Ae_2 = e_1 + 5e_2$

$$\begin{bmatrix} \boxed{7} & 0 & 0 & 0 \\ 0 & \boxed{3} & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$e_1 \xmapsto{A-7I} 0$ ($e_1$ eigenvector $\lambda = 7$)

$e_2 \xmapsto{A-3I} 0$ ($e_2$ " " $\lambda = 3$)

$e_3 \xmapsto{A-5I} 0$ ($e_3$ " " $\lambda = 5$)

$e_4 \xmapsto{A-5I} e_3 \xmapsto{A-5I} 0$ ($e_4$ 2-step e-vector)

$\lambda = 5$

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$e_1 \xmapsto{A-3} 0$

$e_2 \xmapsto{A-3} e_1 \xmapsto{A-3} 0$

$e_4 \xmapsto{A-7} e_3 \xmapsto{A-7} 0$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \curvearrowright \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{3} & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$e_1 \xmapsto{A-3} 0$ ✓

$e_2 \xmapsto{A-3} 0$ ✓

$e_4 \xmapsto{A-3} e_3 \xmapsto{A-3} e_2 \xmapsto{A-3} 0$

(3step $\lambda=3$) (2step w/ $\lambda=3$)

# Sec 7.4/5 "Repeated eigenvalues"

$\boxed{X' = \underline{A}X} + \cancel{X}$ $\quad \cancel{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} + \cancel{\begin{bmatrix} \cos t \\ t \end{bmatrix}}$

$\underline{Ex}$: $A = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix}$.

$\underline{\text{Step 1}}$: find eigenvalues $\lambda$, via $\det(A - \lambda I)$

$\ldots = (\lambda-5)(\lambda-3)^2 = 0$

$\lambda = 5$ alg. mult 1

$\lambda = 3$ alg. mult. 2.

$\underline{\text{Step 2}}$ Compute eigenvectors

$\underline{\lambda = 5}$ $\quad \text{null}(A - 5I) = E_5 = \cdots = \left\{ s\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} : s \in \mathbb{R} \right\}$

pick out $\underline{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, so $\underline{a}$ basis solution

for $X' = AX$ is $\underline{X_1} = e^{5t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

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$\underline{\lambda = 3}$ $\quad \text{null}(A - 3I) = E_3 = \cdots = \left\{ s\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$.

$\underline{v_2} = \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}$ and $\underline{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are two lin. indep

eigenvectors of $\underline{A}$ with $\lambda = 3$.

so $\underline{X_2} = e^{3t}\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}$ $\quad \underline{X_3} = e^{3t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are lin. indep

solutions for $X' = AX$.

$X_1, X_2, X_3$ are LI solutions for system, so general solution

is $X = c_1 X_1 + c_2 X_2 + c_3 X_3$.

$c_1 e^{5t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \ldots$

**Def**: In example, $\lambda = 5$ had alg. mult 1 $\Rightarrow$ $\lambda=5$ defect $d = 1 - 1 = 0$

and $\lambda = 5$ had geom. mult. 1

$\dim(\text{null}(A-\lambda I))$

$\lambda = 3$ had alg. mult. 2

$\lambda = 3$ " geom mult. 2 $\Longrightarrow$ $\lambda = 3$ has defect $2 - 2 = 0$

$\lambda = 5$, and $\lambda = 3$ had defect 0, so they are "<u>complete</u>".

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<u>Ex</u> Compute gen solution for

$$X' = \begin{bmatrix} 5 & 7 \\ 0 & 5 \end{bmatrix} X$$

<u>Step 1</u> $\det(A - \lambda I) = (5-\lambda)^2 = 0$

$\lambda = 5, 5$ alg. mult. 2.

<u>Step 2</u> $\underline{\lambda = 5}$ $\text{null}(A - 5I) \Longleftrightarrow$ solve $\begin{bmatrix} 0 & \boxed{7} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$b = 0$
a free

so $\text{null}(A - 5I) = \left\{ a\begin{bmatrix} 1 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}$

pick out $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, make solution $X_1 = e^{5t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

How to make up for $X_2$? $\lambda = 5$ has defect $d = 2 - 1 = 1$

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$$\underline{X}' = \begin{bmatrix} 5 & 7 \\ 0 & 5 \end{bmatrix} \underline{X}, \quad \underline{X}_1 = \underbrace{e^{5t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e^{\lambda t} \underline{v_1}} \quad \left( \underline{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

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Need solution $\underline{X}_2$ for $\underline{X}' = A\underline{X}$.

- TRY $\underline{X}_2 = e^{\lambda t}(t\underline{v_1})$ ... this FAILS
- TRY $\underline{X}_2 = e^{\lambda t}(t\underline{v_1} + \underline{v_2})$ (don't know $\underline{v_2}$ yet)

Plug in $\underline{X}_2' = A\underline{X}_2$, cancel everything we can...

left over is $A\underline{v_2} = \underline{v_1} + \lambda \underline{v_2} \rightarrow (A - \lambda I)\underline{v_2} = \underline{v_1}$

- So then $\underline{X}_2 = e^{\lambda t}(t\underline{v_1} + \underline{v_2})$ WILL work as a solution IF

$$\longrightarrow \boxed{(A - \lambda I)\underline{v_2} = \underline{v_1}}$$

$\underline{v_1}$ genuine eigenvect $\Longrightarrow$ $(A - \lambda I)\underline{v_1} = 0$

$(A - 5I)\underline{v_1} = 0$

$(A - 5I)\underline{v_2} = \underline{v_1} \Longrightarrow (A-5I)(A-5I)\underline{v_2} = (A-5I)\underline{v_1} = 0$

$(A - 5I)^2 \underline{v_2} = 0$

($\underline{v_2}$ 2-step eigenvector)

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We want $\underline{v_2}$ to satisfy $(A - 5I)\underline{v_2} = \underline{v_1}$

$$\begin{bmatrix} 0 & \boxed{7} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

a free

$7b = 1 \quad b = 1/7$.

Choices for $\underline{v_2}$ are $\begin{bmatrix} a \\ 1/7 \end{bmatrix}$ (a free), pick $\underline{v_2} = \begin{bmatrix} 0 \\ 1/7 \end{bmatrix}$

$$\underline{X}_2 = e^{5t}(t\underline{v_1} + \underline{v_2}) = e^{5t}\left( \begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/7 \end{bmatrix} \right) = e^{5t} \begin{bmatrix} t \\ 1/7 \end{bmatrix}$$

so gen solution

$$\underline{X} = c_1 \underline{X}_1 + c_2 \underline{X}_2 = c_1 e^{5t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} t \\ 1/7 \end{bmatrix} = e^{5t} \begin{bmatrix} c_1 + c_2 t \\ c_2 \cdot 1/7 \end{bmatrix}$$