Sec 7.4/5 "Repeated eigenvalues"

$\boxed{\mathbf{X}' = A\mathbf{X}} + \cancel{\mathbf{X}}$ $\quad \cancel{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} + \cancel{\begin{bmatrix} \cos t \\ t \end{bmatrix}}$

Ex: $A = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix}$.

Step 1: find eigenvalues λ, via $\det(A - \lambda I)$

$$\cdots = (\lambda - 5)(\lambda - 3)^2 = 0$$

$\lambda = 5$ alg. mult 1

$\lambda = 3$ alg. mult. 2.

Step 2 Compute eigenvectors

$\lambda = 5$ $\quad \text{null}(A - 5I) = E_5 = \cdots = \left\{ s\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} : s \in \mathbb{R} \right\}$

pick out $\underline{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, so a basis solution

for $\mathbf{X}' = A\mathbf{X}$ is $\mathbf{X}_1 = e^{5t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

$\lambda = 3$ $\quad \text{null}(A - 3I) = E_3 = \cdots = \left\{ s\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$.

$\underline{v_2} = \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}$ and $\underline{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are two lin. indep

eigenvectors of $\underline{A}$ with $\lambda = 3$.

so $\mathbf{X}_2 = e^{3t}\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}$ $\quad \mathbf{X}_3 = e^{3t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are lin. indep

solutions for $\mathbf{X}' = A\mathbf{X}$.

$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are LI solutions for system, so general solution

is $\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + c_3\mathbf{X}_3$.

$$c_1 e^{5t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \cdots$$

Def: In example, $\lambda=5$ had alg. mult 1 $\Rightarrow$ $\lambda=5$ defect $d = 1 - 1 = 0$

and $\lambda=5$ had geom. mult. 1

$\dim(\text{null}(A-\lambda I))$

$\lambda=3$ had alg. mult. 2

$\lambda=3$ " geom mult. 2 $\Longrightarrow$ $\lambda=3$ has defect $2-2=0$

$\lambda=5$, and $\lambda=3$ had defect 0, so they are "complete".

Ex Compute gen solution for

$$X' = \begin{bmatrix} 5 & 7 \\ 0 & 5 \end{bmatrix} X$$

Step 1 $\det(A-\lambda I) = (5-\lambda)^2 = 0$

$\lambda = 5, 5$ alg. mult. 2.

Step 2 $\lambda=5$ $\text{null}(A-5I) \Longrightarrow$ solve $\begin{bmatrix} 0 & \boxed{7} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$b=0$
a free

so $\text{null}(A-5I) = \left\{ a\begin{bmatrix} 1 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}$

pick out $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, make solution $X_1 = e^{5t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

How to make up for X_2? $\lambda=5$ has defect $d = 2-1 = 1$

$$X' = \begin{bmatrix} 5 & 7 \\ 0 & 5 \end{bmatrix} X, \quad \underbrace{X_1 = e^{5t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e^{\lambda t} v_1} \quad \left(v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

Need solution X_2 for $X' = AX$.

- TRY $X_2 = e^{\lambda t}(t v_1)$ ... this FAILS
- TRY $X_2 = e^{\lambda t}(t v_1 + v_2)$ (don't know v_2 yet)

Plug in $X_2' = AX_2$, cancel everything we can...

left over is $A v_2 = v_1 + \lambda v_2 \rightarrow (A - \lambda I) v_2 = v_1$

- So then $X_2 = e^{\lambda t}(t v_1 + v_2)$ WILL work as a solution IF

$$\longrightarrow \boxed{(A - \lambda I) v_2 = v_1}$$

v_1 genuine eigenvect $\Longrightarrow$ $(A - \lambda I) v_1 = 0$

$(A - 5I) v_1 = 0$

$(A - 5I) v_2 = v_1 \Longrightarrow (A-5I)(A-5I) v_2 = (A-5I) v_1 = 0$

$(A - 5I)^2 v_2 = 0$

(v_2 2-step eigenvector)

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we want v_2 to satisfy $(A - 5I) v_2 = v_1$

$$\begin{bmatrix} 0 & \boxed{7} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

a free

$7b = 1 \quad b = 1/7$

Choices for v_2 are $\begin{bmatrix} a \\ 1/7 \end{bmatrix}$ (a free), pick $v_2 = \begin{bmatrix} 0 \\ 1/7 \end{bmatrix}$

$$X_2 = e^{5t}(t v_1 + v_2) = e^{5t}\left(\begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/7 \end{bmatrix} \right) = e^{5t}\begin{bmatrix} t \\ 1/7 \end{bmatrix}$$

so gen solution

$$X = c_1 X_1 + c_2 X_2 = c_1 e^{5t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} t \\ 1/7 \end{bmatrix} = e^{5t}\begin{bmatrix} c_1 + c_2 t \\ c_2 \cdot 1/7 \end{bmatrix}$$