Phase plane portraits

Draw solution curves for system $\underline{X}' = \underline{A}\underline{X}$ ($A_{2\times 2}$)

Cases where gen solution has form

$$\underline{X}(t) = c_1 e^{\lambda_1 t}\underline{v_1} + c_2 e^{\lambda_2 t}\underline{v_2}. \quad (c_1\underline{X}_1 + c_2\underline{X}_2)$$

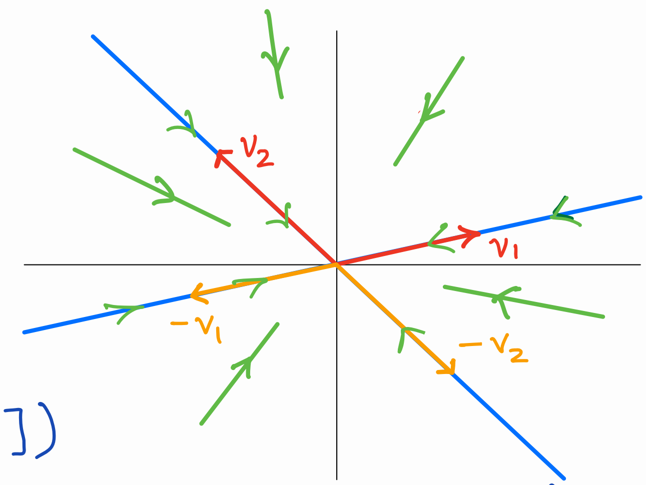
note $\underline{X}(0) = c_1\underline{v_1} + c_2\underline{v_2}$ (initial position)

we assume t can range over all of $(-\infty, \infty)$

Case: repeated $\lambda \in \mathbb{R}$; no defect.

$$\underline{X}(t) = e^{\lambda t}(c_1\underline{v_1} + c_2\underline{v_2})$$

- $\lambda < 0 \longrightarrow$ "proper nodal sink"
- $\lambda > 0 \longrightarrow$ "proper nodal source"
- $\lambda = 0 \longrightarrow$ all points stationary (only possible if $\underline{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$)



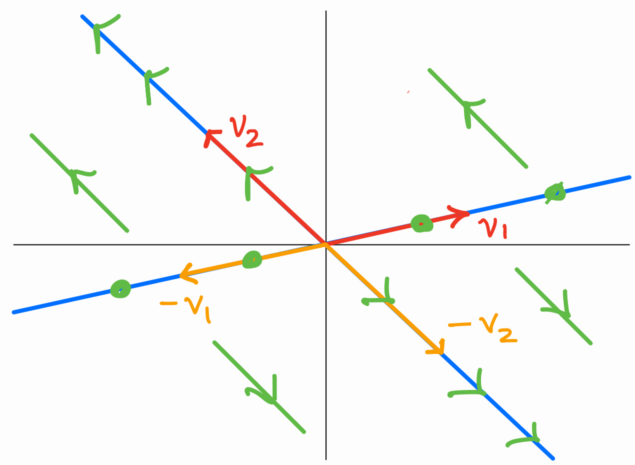
(proper nodal sink)

Case: $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_2 \in \mathbb{R}$.

$$\underline{X}(t) = c_1\underline{v_1} + c_2 e^{\lambda_2 t}\underline{v_2}$$

"parallel lines" (in/out depending on $\lambda_2 < 0$ or > 0)

(lines perpindicular to $\underline{v_1}$ line)



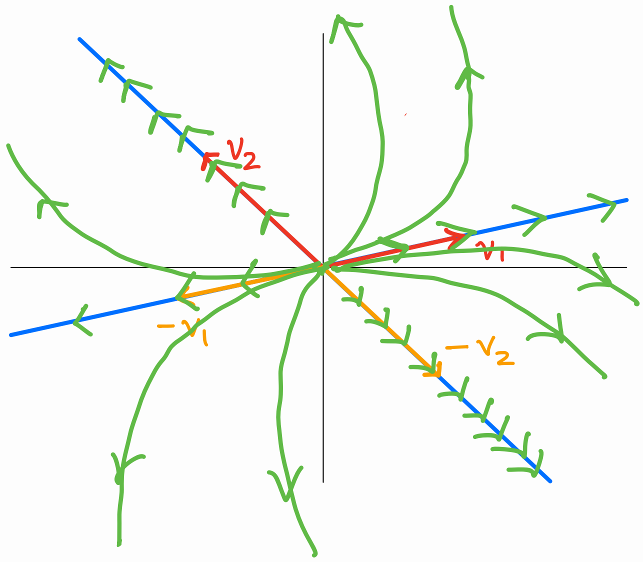
Case : $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$, same sign

$$\underline{X}(t) = c_1 e^{\lambda_1 t} \underline{v_1} + c_2 e^{\lambda_2 t} \underline{v_2}$$

"improper nodal source/sink"

Ex : $0 < \lambda_1 < \lambda_2$

λ_2 larger (absolute value) $\Rightarrow$ stronger "acceleration" along $\underline{v_2}$

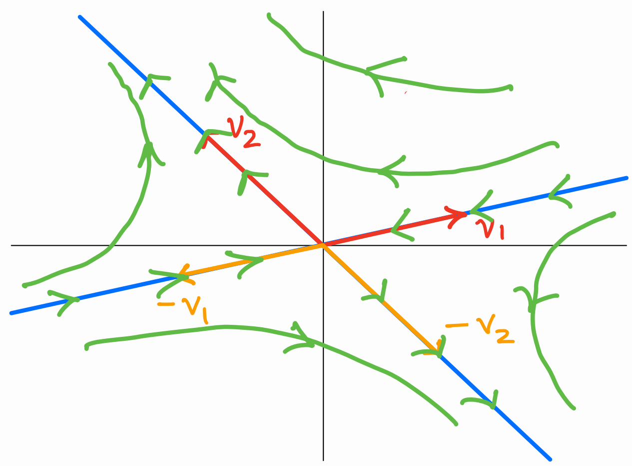


Case $\lambda_1 \neq \lambda_2$, $\lambda_1 < 0 < \lambda_2$

$$\underline{X} = c_1 e^{\lambda_1 t} \underline{v_1} + c_2 e^{\lambda_2 t} \underline{v_2}$$

"saddle point"

Ex : $\lambda_1 < 0 < \lambda_2$ $\longrightarrow$

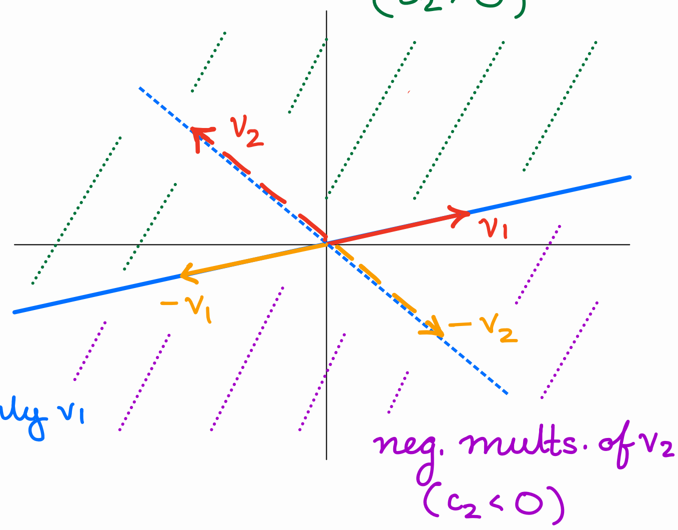


Cases of repeated λ with defect.

v_1 is genuine eigenvector

v_2 is generalized " " (2-step),

$$\mathbf{X}(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t}(t v_1 + v_2)$$
$$= (c_1 + c_2 t) e^{\lambda t} v_1 + c_2 e^{\lambda t} v_2$$



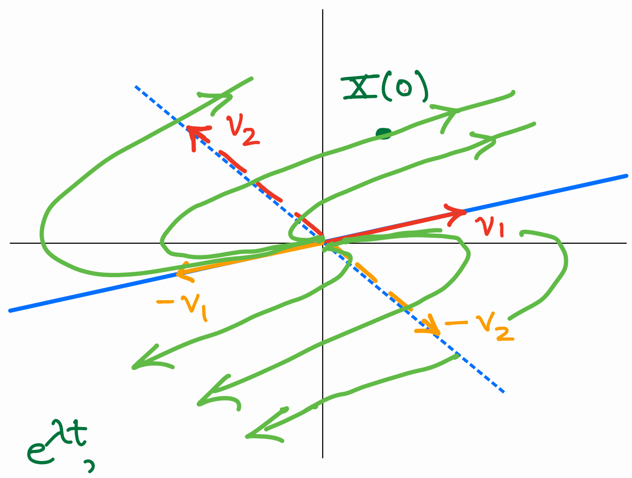
These make "<u>improper nodal source/sinks</u>" $\boxed{\text{if } \lambda \neq 0}$

trajectory over time

ex: $\lambda > 0$, $c_1 = 1$, $c_2 = 1/2$

$$\mathbf{X}(t) = \left(1 + \frac{t}{2}\right) e^{\lambda t} \underline{v_1} + \frac{1}{2} e^{\lambda t} \underline{v_2}$$

$$\mathbf{X}(0) = \underline{v_1} + \frac{1}{2} \underline{v_2}$$



As $|t| \to \infty$, $t e^{\lambda t}$ dominates over $e^{\lambda t}$,

so as $t \to \pm\infty$,

$$\mathbf{X}(t) \approx \frac{1}{2} t e^{\lambda t} \underline{v_1} \quad (\text{parallel to } \underline{v_1} \text{ line})$$

- $t \approx \infty$, "heading" is $+\underline{v_1}$
- $t \approx -\infty$, "heading" is $-\underline{v_1}$ ...

OTOH, if $t \approx -\infty$ then $t e^{\lambda t} \underline{v_1} \approx 0$, so $\mathbf{X}$ at origin

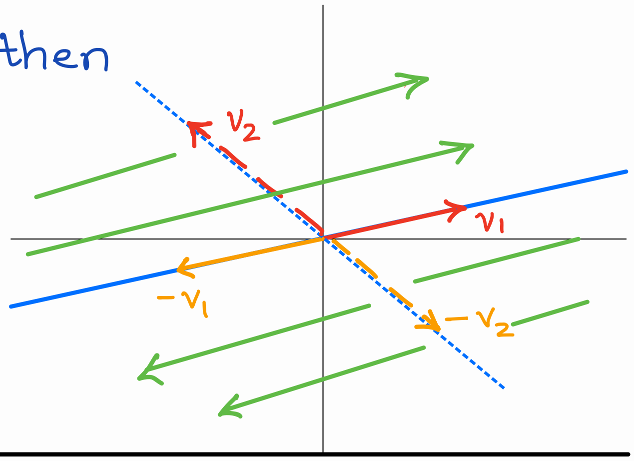
between $-\infty$, ∞ we make "u-turn"

If $\lambda = 0$ is repeated, defect > 0, then

$$\underline{X}(t) = (c_1 + c_2 t) v_1 + c_2 v_2$$

"parallel lines" again,

but they are parallel to $\underline{v_1}$



Case of complex conjugate pair $\lambda = p \pm qi$

Say $\underline{v}$ is eigenvector for $\lambda = p + qi$, and

$\underline{v} = \underline{a} + \underline{b}\, i$ (real & imaginary parts)

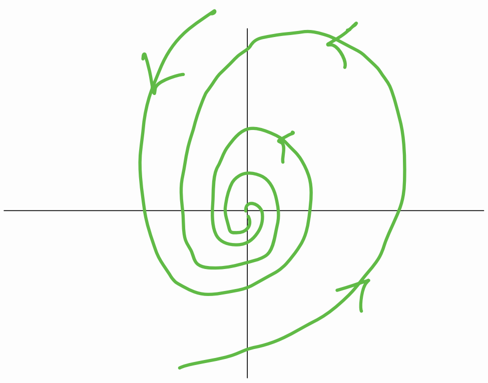
Then basis solutions are

$$\underline{X}_1 = \mathrm{Re}(e^{\lambda t} \underline{v}) = \cdots = e^{pt}(\underline{a} \cos(qt) - \underline{b} \sin(qt))$$

$$\underline{X}_2 = \mathrm{Im}(e^{\lambda t} \underline{v}) = \cdots = e^{pt}(\underline{b} \cos(qt) + \underline{a} \sin(qt))$$

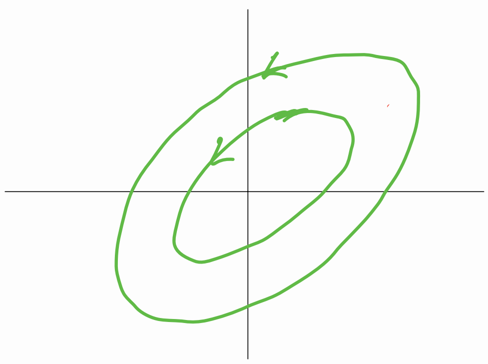
gen solution $\underline{X} = c_1 \underline{X}_1 + c_2 \underline{X}_2$

- much harder to plot eigenvectors (recommend avoid)



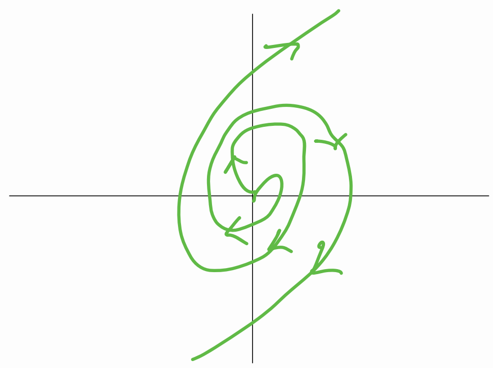
$p < 0$

"spiral sink"



$p = 0$

"center"



$p > 0$

"spiral source"

☆ orientation (cw/ccw) depends on example (use velocity $\underline{X}'$ to check)