Sec 2.1 Population Models

★ For Monday 2/8, Read book section 3.1

$P(t) =$ Population

- Exponential growth model $\frac{dP}{dt} = kP$

"a more realistic model" is that

$$\frac{dP}{dt} = \beta P - \delta P$$

& $\beta =$ "birth parameter" $= \frac{\text{births/unit}}{\text{time}}$.

$\delta = \frac{\text{deaths/unit}}{\text{time}}$

Empirically, a "good/realistic" model for populations

has $\beta(t) = \beta_0 - \beta_1 \cdot P(t)$

$\delta(t) = \delta_0$

($\beta_0, \beta_1, \delta_0$ const.)

then $\frac{dP}{dt} = \beta_0 P - \beta_1 \cdot P^2 - \delta_0 P$

$$\frac{dP}{dt} = aP - bP^2$$

$a = \beta_0 - \delta_0$

$b = \beta_1$

The model/equation

$$\boxed{\frac{dP}{dt} = aP - bP^2}$$

(w/ a, b constant and $a, b > 0$)

is the "Logistic equation"

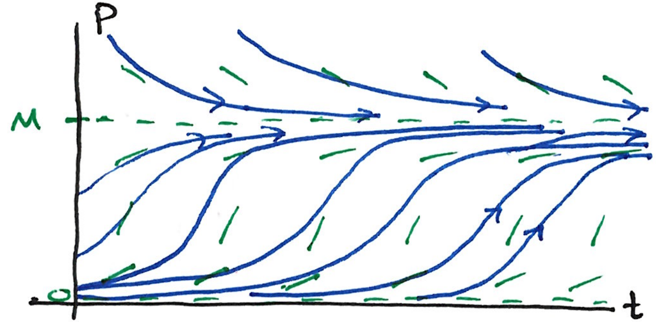
$$\frac{dP}{dt} = aP - bP^2 \quad (\text{Logistic Eq})$$

$$\boxed{\frac{dP}{dt} = bP\left(\frac{a}{b} - P\right) = kP(M - P)}$$

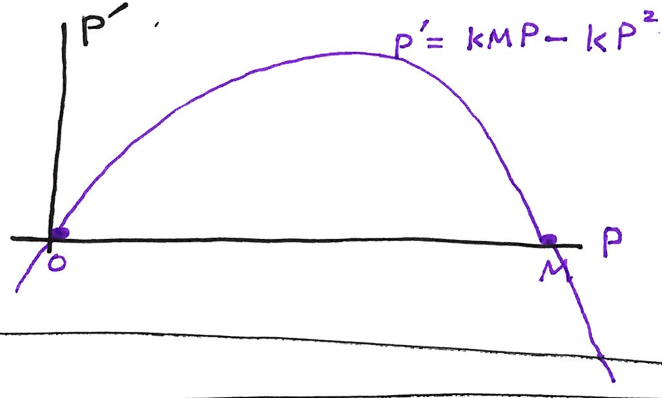
$k = b$
$M = a/b$
(Logistic Eq alt. form)

<u>Observe</u>: $P' = kP(M-P)$

- $P' = 0$ if $P = 0$, $P = M$
- $P' > 0$ if $0 < P < M$
- $P' < 0$ if $P > M$



- M is called the "limiting population" or "carrying capacity".



Logistic equation is $\boxed{\dfrac{dP}{dt} = aP - bP^2}$, $a, b > 0$ (const.)

alt./equivalently it is $\boxed{\dfrac{dP}{dt} = kP(M-P)}$ $k, M > 0$ (const)

Separate variables and use partial fractions to solve the initial value problem. Use either the exact solution or a computer-generated slope field to sketch the graphs of several solutions of the given differential equation, and highlight the indicated particular solution.

$$\frac{dx}{dt} = 8\left(x - x^2\right),\ x(0) = 2$$

separable

x(t) = ☐

$$\frac{dx}{x - x^2} = 8dt \longrightarrow \int \frac{dx}{x(1-x)} = \int 8dt$$

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x} \Longrightarrow 1 = A(1-x) + Bx$$

$$\underline{0x} + \underline{1} = \underline{A} + \underline{(B-A)}x$$

$$\Longrightarrow A = 1$$

$$B - A = 0 \Longrightarrow B = A = 1.\text{, so } \frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

$$\int \frac{dx}{x} + \int \frac{dx}{1-x} = 8t + C$$

$$\ln|x| - \ln|1-x| = 8t + C \qquad \ln(a) - \ln(b) = \ln(a/b)$$

$$\ln\left|\frac{x}{1-x}\right| = 8t + C$$

$$\left|\frac{x}{1-x}\right| = Ce^{8t} \Longrightarrow \frac{x}{1-x} = Ce^{8t}$$

have data point $(t=0, x=2)$

$$\frac{2}{1-2} = Ce^0 \Longrightarrow C = -2$$

result for now is $\dfrac{x}{1-x} = -2e^{8t}$. Want $x(t) = \ldots$

$$\frac{x}{1-x} = -2e^{8t} \implies x = (1-x)\cdot(-2)e^{8t}$$

$$x = -2e^{8t} + 2xe^{8t}$$

$$x(1-2e^{8t}) = -2e^{8t}$$

$$x = \frac{+2e^{8t}}{-1+2e^{8t}} = \frac{2}{(-1e^{-8t}+2)}\,\frac{e^{8t}}{e^{8t}}$$

$$\boxed{x(t) = \frac{2}{2-e^{-8t}}}$$

$$\left\{ \frac{dx}{dt} = 8(x-x^2),\quad x(0)=2 \right\}$$

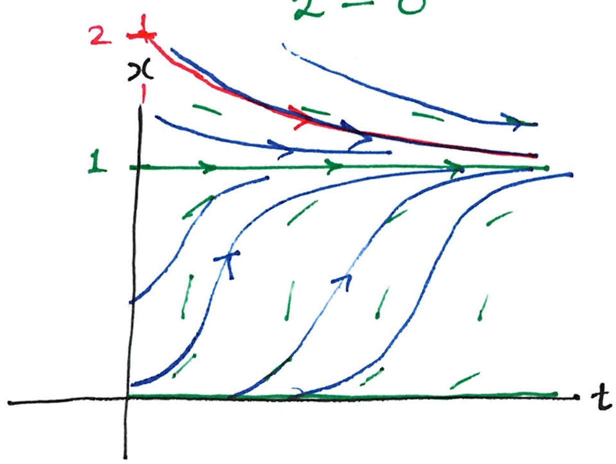
$$\frac{dx}{dt} = 8x(1-x)$$

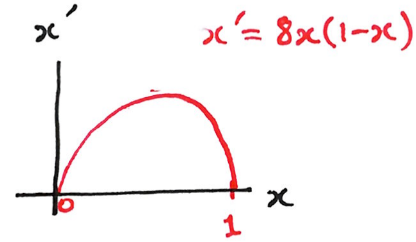
$$M = 1$$

- $x(0) = \frac{2}{2-1} = 2$

- as $t \to \infty$, $e^{-8t} \to 0$, so that leaves us with

$$x(t) \approx \frac{2}{2-0} = 1$$





$$\text{pop change rate} = \frac{\text{(total) birth}}{\text{time}} - \frac{\text{(total) death}}{\text{time}}$$

Consider a rabbit population P(t) satisfying the logistic equation $\frac{dP}{dt} = aP - bP^2$, where $\boxed{B = aP \text{ is the time rate at which births}}$ occur and $\boxed{D = bP^2 \text{ is the rate at which deaths occur.}}$ If the initial population is 240 rabbits and there are 6 births per month and 12 deaths per month occurring at time t = 0, how many months does it take for P(t) to reach 110% of the limiting population M?

☐ months (Round to two decimal places as needed.)

B = birth/~~time~~ @ time t

D = death/time @ " "

$P(0) = 240$

$$\left\{ \frac{dP}{dt} = aP - bP^2 \right\}$$

$$@\ t=0 \quad B = aP = 6 \implies 6 = aP(0) = a \cdot 240$$

$$\implies \boxed{a = \frac{6}{240} = \frac{1}{40}}$$

$$@\ t=0 \quad D = bP^2 = 12 \implies 12 = b\,P(0)^2 = \quad 12 = b \cdot 240 \cdot 240$$

$$b = \frac{12}{240} \cdot \frac{1}{240} = \frac{1}{20} \cdot \frac{1}{240} = \boxed{\frac{1}{4800} = b}$$

$$\frac{dP}{dt} = \frac{1}{40}P - \frac{1}{4800}P^2 = aP - bP^2 = \text{"}\ bP\left(\frac{a}{b} - P\right)$$

$$M = a/b = \frac{1}{40} \cdot \frac{4800}{1} = \frac{12 \cdot 100}{10} = \boxed{120}$$

Limiting pop is 120, equation is

$$\left\{ \frac{dP}{dt} = \frac{1}{4800} P(120 - P), \quad \text{(logistic equation)} \quad P(0) = 240 \right\}$$

$$\left\{ \frac{dP}{dt} = kP(120-P), \quad P(0) = 240 \right\} \qquad \left(k = \frac{1}{4800} \right)$$

separable, (alternatively, $\frac{dP}{dt} = 120kP - kP^2$

$$\frac{dP}{dt} - 120 \cdot kP = -kP^2 \quad \left(\text{Bernoulli}, \; n=2 \right) \Big)$$

$$\int \frac{dP}{P(120-P)} = \int k\,dt$$

$$\frac{1}{P(120-P)} = \frac{A}{P} + \frac{B}{120-P}$$

$$1 = A(120-P) + BP$$

$$1 = 120A + (B-A)P$$

$$\Rightarrow \; 120A = 1 \Rightarrow A = \frac{1}{120}$$

$$B - A = 0 \Rightarrow B = A = \frac{1}{120}$$

$$\frac{1}{120}\int \frac{dP}{P} + \frac{1}{120}\int \frac{dP}{120-P} = kt + C$$

$$\frac{1}{120}\left(\ln|P| - \ln|120-P| \right) = kt + C$$

$$\ln\left| \frac{P}{120-P} \right| = 120kt + C$$

$$\frac{P}{120-P} = Ce^{120kt}$$

know $(t=0, \; P=240)$

$$\frac{240}{120-240} = C = \frac{240}{-120} = -2$$

$$\frac{P}{120-P} = -2e^{120kt}$$

$$P = -240e^{120kt} + 2Pe^{120kt}$$

$$P \cdot \left(1 - 2e^{120kt}\right) = -240e^{120kt}$$

$$P = \frac{-240e^{120kt}}{1-2e^{120kt}} = \frac{-240 \cdot \cancel{e^{120kt}}}{\left(e^{-120kt} - 2\right) \cdot \cancel{e^{120kt}}}$$

$$\boxed{P = \frac{240}{2 - e^{-120kt}}} \qquad \left(k = \frac{1}{4800} \right)$$

$$\boxed{P = \frac{240}{2 - e^{-120kt}} = \frac{240}{2 - e^{-t/40}}}$$

$120k = \frac{120}{4800} = \frac{1}{40}$

$P(0) = \frac{240}{2-1} = 240$ ✓

$\lim_{t\to\infty} P(t) = \frac{240}{2} = 120 = M$ ✓

$$\left\{ \frac{dP}{dt} = \frac{1}{4800} P(120-P) \right\}$$

$P(0) = 240$

How long until $P(t) = M\cdot(1.10) = 120\cdot(1.1) = 132$. ?

$$132 = \frac{240}{2 - e^{-t/40}} \implies 2 - e^{-t/40} = \frac{240}{132}$$

$$2 - \frac{240}{132} = e^{-t/40}$$

$$t = (-40)\ln\left(2 - \frac{240}{132}\right)$$

$t \approx 68$ months