Linear ODEs Summary

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Linear ODEs

Definition. An **order-n** (or "n-th order") **linear ODE** involving the function $y(x)$ is an ODE of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x). \tag{1}$$

- If you say "n-th order", implicitly $a_n \neq 0$, because if not then the ODE would be some lower order.
- Often $a_0, a_1, \ldots, a_n$ are constants, but *not always*—sometimes they *depend on x*.
- The linear ODE is homogeneous if the right-hand side $f(x)$ is just 0.

Remark. ("L-operator") When working with a specific linear ODE, it's often convenient to abbreviate the left-hand side as the result of a template/operator "L" applied to the function y; we write this

$$L[y] \longleftrightarrow a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y,$$

using whatever values of a_i we are given.

Note: Thus, in the following notes, for brevity we'll associate/understand

$$L[y] = 0 \quad \longleftrightarrow \quad \text{homogeneous linear ODE},$$
$$L[y] = f(x) \quad \longleftrightarrow \quad \text{nonhomogeneous linear ODE (if } f \neq 0\text{)}.$$

Again, what the template/operator $L[y]$ actually *is* depends on the problem you're solving.

"Format" of solutions for linear ODEs

(1) (**Homogeneous eq:** $L[y] = 0$) To describe the "full" general solution of a homogeneous n-th order linear ODE $L[y] = 0$, one requires n linearly independent "basis solutions" $\{y_1, y_2, \ldots, y_n\}$ (solutions of eq. $L[y] = 0$).

Once you have those basis solutions, the general solution of $L[y] = 0$ is

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

(2) (**Nonhomogeneous eq:** $L[y] = f$) To describe the "full" general solution of a nonhomogeneous n-th order linear ODE $L[y] = f$, you need "two" parts:

(a) The *general* solution y_c of the associated homogeneous eq $L[y] = 0$ (same L); naturally, to get y_c you have to get the n basis solutions for $L[y] = 0$.

(b) A *particular* solution y_p of the equation $L[y_p] = f$.

Once you have both, the general solution for $L[y] = f$ is

$$y = y_c + y_p.$$

Note on nonhomogeneous equation solution: In finding the general solution for $L[y] = f$, it may seem odd to keep going after you have found y_p for $L[y_p] = f$. However, because

$$L[y_c + y_p] = L[y_c] + L[y_p] = 0 + f = f,$$

adding-on any y_c which makes $L[y_c] = 0$ creates another solution to $L[\cdot] = f$. Thus, to describe the "full set" of solutions for $L[y] = f$, we need to include that "y_c+" with y_p.

Linear independence; the Wronskian.

Definition. A set of functions $\{y_1, y_2, \ldots, y_n\}$ (all defined on some common domain) is linearly independent (LI) if the linear combination

$$c_1 y_1 + \cdots c_n y_n \equiv 0 \qquad \text{(meaning the combination is the constant-0 function)}$$

is possible *only* with the "trivial combination" where $(c_1, \ldots, c_n) = (0, \ldots, 0)$. If there is some nontrivial combination making the constant-0 function, then the functions y_i are linearly dependent (LD).

Definition. Given a set of functions $\{y_1, \ldots, y_n\}$, their Wronskian $W(y_1, \ldots, y_n)$ is a function (of x) defined according to the following determinant pattern:

Two functions $\{y_1, y_2\}$: $$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 \cdot y_2' - y_2 \cdot y_1'.$$

Three functions $\{y_1, y_2, y_3\}$: $$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix},$$
$$= y_1 \cdot (y_2' \cdot y_3'' - y_2'' \cdot y_3') - y_2 \cdot (\cdots) + y_3 \cdot (\cdots)$$

and then for four functions it is a 4×4 matrix (with y_i''' in the last row), and so forth.

$$W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$$

$$\begin{aligned} W(x, \cos x, \sin x) &= \begin{vmatrix} x & \cos x & \sin x \\ 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} \\ &= x(\sin^2 x + \cos^2 x) - 1(-\cos(x)\sin(x) + \sin(x)\cos(x)) + 0 \\ &= x \end{aligned}$$

Theorem. *Suppose that $\{y_1, \ldots, y_n\}$ are all solutions to a certain n-th order* linear ODE, and form/compute their Wronskian $W(y_1, \ldots, y_n)(x)$. Then one of two things can happen:*

(1) *If the Wronskian W is not constantly 0 on its domain (this is the same domain as the y_i), then the functions y_i are LI.*

(2) *If the Wronskian W is 0 everywhere on its domain, then the functions y_i are LD.*

*Though a technicality we rarely have to deal with, it is important here that the number of functions matches the order of the ODE.

Theorem. *An n-th order linear, homogeneous ODE always requires and has*[†] *n linearly independent basis solutions to describe its general solution.*

Solutions to homogenous eqs with constant coefficients

Note: In this section, we'll only focus on ODEs

$$L[y] = 0, \qquad \longleftrightarrow \qquad y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 = 0,$$

where all a_i are constant.

Definition. If $L[y] = 0$ is a linear homogeneous ODE with const. coeffs., aka

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 = 0$$

then the characteristic polynomial of the ODE is

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0.$$

The characteristic equation is simply

$$\text{“}(\textit{char. poly.}) = 0\text{”}, \qquad \text{aka} \qquad r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

Roots of the char. poly./eqn.

Theorem. *Suppose that $y(x)$ is a complex-valued solution to the homogeneous linear ODE $L[y] = 0$, meaning that y has both a real and an imaginary part, i.e., $y(x) = a(x) + b(x) \cdot i$, where $a(x)$ and $b(x)$ are some functions. Then both $a(x)$ and $b(x)$ are solutions of the same equation. Succinctly,*

$$L[y] = 0 \quad \textit{and} \quad y(x) = a(x) + b(x)i \qquad \Longrightarrow \qquad L[a] = 0 \quad \textit{and} \quad L[b] = 0,$$

or

$$L[y] = 0 \qquad \Longrightarrow \qquad L[\mathrm{Re}(y)] = 0 \quad \textit{and} \quad L[\mathrm{Im}(y)] = 0,$$

"Proof". Really this amounts to observing that

$$L[y] = L[a + bi] = L[a] + i \cdot L[b].$$

■

The number of times r occurs as a root of the char. poly. is the "multiplicity of r." We treat roots differently based on if they are real or complex numbers.

(1) (Real roots) Suppose that r is a root of the char. poly. of a homog. lin. ODE with const. coeffs., and that r has multiplicity k as a root. Then the functions

$$\{e^{rx}, xe^{rx}, x^2 e^{rx}, \ldots, x^{k-1}e^{rx}\} \qquad \text{are } k \text{ lin. indep. solutions to the ODE,}$$

so they should be included in the set of "basis solutions".

[†]Of course, there can be many different choices of sets of n basis solutions—any set of n LI solutions is valid to make a basis.

(2) (Complex roots) Suppose that the "conjugate pair" of roots $r = a \pm bi$ occurs k times as roots of the char. poly.[‡] Then

$$\Big\{ e^{(a+bi)x},\ xe^{(a+bi)x},\ \ldots,\ x^{k-1}e^{(a+bi)x},$$
$$e^{(a-bi)x},\ xe^{(a-bi)x},\ \ldots,\ x^{k-1}e^{(a-bi)x} \Big\}$$

is indeed a set of $2k$ lin. indep. solutions for the ODE.

However, we prefer to make $2k$ solutions by taking the real and imaginary parts of the "first half" of the solutions; i.e.

$$y = e^{(a+bi)x} = e^{ax}\cos(bx) + ie^{ax}\sin(bx) \quad \xrightarrow{\text{Re/Im}} \quad y_1 = e^{ax}\cos(bx), \quad \tilde{y}_1 = e^{ax}\sin(bx),$$
$$y = xe^{(a+bi)x} \quad \xrightarrow{\text{Re/Im}} \quad y_2 = xe^{ax}\cos(bx), \quad \tilde{y}_2 = xe^{ax}\sin(bx),$$

and so forth, gives us $2k$ lin. indep. solutions to the equation, namely

$$\Big\{ e^{ax}\cos(bx),\ xe^{ax}\cos(bx),\ \ldots,\ x^{k-1}e^{ax}\cos(bx),$$
$$e^{ax}\sin(bx),\ xe^{ax}\sin(bx),\ \ldots,\ x^{k-1}e^{ax}\sin(bx) \Big\}.$$

In either case, we include these basis functions in making our full general solution for the ODE.

Spring oscillations

- The "damped spring equation" for a mass of m (kg) attached to a spring with constant k (N/m) and a damper/shock with damping constant $c > 0$ is
$$mx'' + cx' + kx = 0.$$
Here x' and x'' are the time t-derivatives of x.
- The roots of the char. poly. of the equation are
$$r = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}.$$
- $\omega_0 = \sqrt{k/m}$ is the "angular frequency".

The nature of the solution(s) $x(t)$ naturally depends on the values of m, c, and k. The four cases are:

(1) ($c = 0$) This is the <u>undamped</u> case, and the roots are

$$r = \pm\frac{i\sqrt{4km}}{2m} = \pm\omega_0 i.$$

Thus, the basis solutions are

$$\big\{ \cos(\omega_0 t),\ \sin(\omega_0 t) \big\}.$$

Note: For the remaining cases, we abbreviate $p = \frac{c}{2m}$.

[‡]This is no different than saying that $a + bi$ has multiplicity k and $a - bi$ also has multiplicity k.

(2) ($c \neq 0$, $c^2 > 4km$) Here we have two distinct, real roots

$$r_1 = -p + \frac{\sqrt{c^2 - 4km}}{2m} \qquad \text{and} \qquad r_2 = -p - \frac{\sqrt{c^2 - 4km}}{2m},$$

and the basis solutions are

$$\left\{e^{r_1 t},\, e^{r_2 t}\right\}.$$

Unpacking the definition of $p = c/2m$, you would see that both $r_1, r_2 < 0$, hence the motion does eventually decay. This is the overdamped case.

(3) ($c \neq 0$, $c^2 = 4km$) Here the square-root expression vanishes, leaving us with root(s)

$$-p + 0, \quad -p - 0, \qquad \text{meaning } -p \text{ is a root with multiplicty 2.}$$

Hence, the basis solutions are

$$\left\{e^{-pt},\, te^{-pt}\right\}.$$

This is the critically damped case.

(4) ($c \neq 0$, $c^2 < 4km$). The quantity in the square-root is negative, giving an imaginary number overall. Thus, defining/writing ω_1 to make

$$-p \pm \frac{\sqrt{c^2 - 4km}}{2m} \qquad \longrightarrow \qquad -p \pm \omega_1 i,$$

the basis solutions are

$$\left\{e^{-pt}\cos(\omega_1 t),\, e^{-pt}\sin(\omega_1 t)\right\}.$$

This is the underdamped case.

Nonhomogenous linear ODEs

A nonhomogenous linear ODE will be an equation

$$L[y] = f \qquad \text{where} \qquad f \neq 0.$$

As we said before, the general solution for such an eq is

$$y = y_c + y_p,$$

where

- y_c is the general solution of the associated eq $L[y] = 0$.
- y_p is any particular solution of the equation $L[y] = f$.

Note: the first step in solving such an equation is always first finding the "complementary function/solution" y_c for $L[y_c] = 0$. Once you have that, your task is to find a particular solution y_p for $L[y_p] = f$. There are two primary methods for this:

Method of undetermined coeffs.; trial functions. For each distinct "piece/part" of the right-hand side $f(x)$, we include a corresponding "part" in our trial function y_p, as exampled:

(1) If f has a degree-d polynomial, then include a degree-d piece in the trial function y_p. For instance,

$$f = 2 - 3x + 5x^2 \quad \Rightarrow \quad \text{include trial piece} \quad A + Bx + Cx^2 \quad \Rightarrow \quad .$$

(2) If f has an exponential e^{kx}, include term(s) Ae^{kx}.

(3) If f has either cos or sin, then include terms for both cos and sin, e.g.,

$$f(x) = 2\cos(3x) \quad \Rightarrow \quad \text{use/include trial piece} \quad A\cos(3x) + B\sin(3x).$$

(4) If f has some "mixed" terms, like x^2e^{5x}, then use similar "mixed" terms in the trial function; for instance,

$$f = x^2e^{5x} \quad \Rightarrow \quad \text{include} \quad Ae^{5x} + Bxe^{5x} + Cx^2e^{5x}, \quad \Rightarrow \quad \text{pieces } \{e^{5x}, xe^{5x}, x^2e^{5x}\}$$

$$f = e^{7x}\cos(2x) \quad \Rightarrow \quad \text{include} \quad Ae^{7x}\cos(2x) + Be^{7x}\sin(2x), \quad \Rightarrow \quad \text{pieces } \{e^{7x}\cos(2x), e^{7x}\sin(2x)\}.$$

Method of undetermined coeffs.; removing "duplication". Again we assume we are we solving $L[y] = f$, and have found y_c for $L[y] = 0$. Let's say that

$$y_c = c_1y_1 + c_2y_2 + \cdots c_ny_n \quad \Longrightarrow \quad \text{pieces/basis solutions } \{y_1, y_2, \ldots, y_n\}.$$

If our trial function has a piece/part that is "duplicated" in the list $\{y_1, \ldots, y_n\}$, then we need to modify our trial y_p to make all of its pieces independet of all the pieces of y_c.

This is perhaps best summarized with example: If

$$y_c = c_1 \cdot 1 + c_2x + c_3\cos(4x) + c_4\sin(4x) \quad \Longrightarrow \quad \text{pieces } \{1, x, \cos(4x), \sin(4x)\},$$

and our initial trial function

$$y_p = A + Bx + C\cos(7x) + D\sin(7x) \quad \Longrightarrow \quad \text{pieces } \{1, x, \cos(7x), \sin(7x)\},$$

then we need to "remove the duplication" of $\{1, x\}$ with those pieces of y_c. Note that we *don't* need to worry about the cos and sin terms, because our y_p terms have different frequencies than the y_c pieces.

Thus, our new trial function should be

$$y_p = Ax^2 + Bx^3 + C\cos(7x) + D\sin(7x) \quad \Longrightarrow \quad \text{pieces } \{x^2, x^3, \cos(7x), \sin(7x)\}.$$

Method of variation of parameters (NOT FOR EXAM 2). Given a second-order linear, nonhomogenous ODE, say $L[y] = f$, again assume that you have found the "complementary function/solution" y_c for $L[y] = 0$. Because the eq is 2nd-order, the general solution y_c will involve two basis solutions y_1 and y_2; therefore of course

$$y_c = c_1y_1 + c_2y_2.$$

The method of "variation of parameters" is used when the right-hand side $f(x)$ is not one of our known "trial" cases, or has a form that changes drastically when you differentiate it (ex: $f = \ln x$ and $f' = x^{-1}$ are quite different in appearance.)

The method has the following steps for finding y_p.

(1) Let $u_1 = u_1(x)$ and $u_2 = u_2(x)$ be two unknown functions, and form the trial function
$$y_p = u_1 y_1 + u_2 y_2.$$

(2) Assuming (simply to narrow our search pool of u_1 and u_2) that
$$u_1' y_1 + u_2' y_2 = 0,$$
we plug in y_p to $L[y_p]$, and obtain the "linear" system of equations
$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0, \\ u_1' y_1' + u_2' y_2' &= f. \end{aligned} \qquad \longleftrightarrow \qquad \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

(3) Noting that the Wronskian $W(y_1, y_2) = y_1 y_2' - y_1' y_2$, we invert the matrix above to solve the system, i.e.,
$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix},$$
and so
$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix} = \begin{bmatrix} -y_2 f/W \\ y_1 f/W \end{bmatrix}.$$

(4) Then we integrate to find that
$$u_1 = \int \frac{-y_2 f}{W}\, dx \qquad \text{and} \qquad u_2 = \int \frac{+y_1 f}{W}\, dx,$$
and use these in $y_p = u_1 y_1 + u_2 y_2$.