Notes on vector spaces

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Linear combinations and spanning sets

(1) A linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a "weighted sum" of the vectors (of course the result is again a vector), as in a sum

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k,$$

where c_i are scalars.

Definition. Given a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, the span of those vectors is the *set* of **all** linear combinations of them. That is,

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \Big\{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k : c_1, \ldots, c_k \in \mathbb{R}\Big\}.$$

If V is some vector space (or W is some subspace), and some list of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ has $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = V$ (or $= W$), then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is said to be a spanning set of V (or of W).

Notes about span:

(1) $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is also sometimes called "*the subspace generated by* $\mathbf{v}_1, \ldots, \mathbf{v}_k$". This is reasonable since "spans are automatically subspaces".

(2) Vector spaces and subspaces will have many different spanning sets, so one should never think spanning sets are "unique".

Linear (in-)dependence (examples for 3 vectors)

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors all of the same size. They could be 2×1, 3×1, etc.; all that matters is they are of equal size.

Definition. Given vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, we consider the tuples of scalars (a, b, c) such that the linear combination

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}.$$

That is, we want to know what the solutions (a, b, c) for the system of equations are.

(1) If the "*trivial solution*" $(a, b, c) = (0, 0, 0)$ is the only solution of the system, then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent (LI).

(2) If there is at least one "*nontrivial solution*" $(a, b, c) \neq (0, 0, 0)$, then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly dependent (LD).

Remark. "$\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are (LI/LD)" can also be said as "$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a (LI/LD) set."

"Equivalent" conditions for LI/LD.

Here "equivalent" means that if the vectors are LI in any one of the following descriptions, they are automatically LI in all the others, and same for LD.

(1) $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a (LI/LD) set if (none/at least one) of the vectors is in the **span** of the others.

(2) The system of equations/vector equation

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}, \quad \text{AKA} \quad \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{0}, \quad \text{AKA} \quad \mathbf{A}\mathbf{x} = \mathbf{0},$$

has (only the *trivial solution* $(a, b, c) = (0, 0, 0)$ / some *nontrivial solutions* $(a, b, c) \neq (0, 0, 0)$).

(3) By definition of span, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a *spanning set* for $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is LI, then it is a *basis* for $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$; if it is LD, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is *not* be a basis for $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

(4) Like in the last item, by definition the set $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for $V = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, and S (is/is not) a basis for V if S is (LI/LD). Then
   (a) If S is a LI set then it is a basis for V, so it must be that $\dim(\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}) = 3$.
   (b) If S is a LD set then it is not a basis for V, so it must be that $\dim(\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}) < 3$. (The dimension cannot be > 3 because 3 vectors can generate *at most* 3 "dimensions").

(5) $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly (LI/LD) set if the determinant*

$$\det(\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}) \qquad (= 0 \ / \neq 0)$$

*We can only do this method if $\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$ is a square matrix. If the matrix isn't square we need to use one of the previous items.

Linear (in-)dependence (more generally)

Definition. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors all of the same size (say $n \times 1$, so they are in $\mathbb{R}^n$), and consider the system of equations/vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

where c_1, c_2, ..., c_k are all unknown scalars.

(1) If the "trivial solution" $(c_1, c_2, \ldots, c_k) = (0, 0, \ldots, 0)$ is the only solution of the system, then $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ are linearly independent (LI).

(2) If there is at least one additional solution $(c_1, c_2, \ldots, c_k) \neq (0, 0, \ldots, 0)$, then $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ are linearly dependent (LD).

Remark. "*The vectors* $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ *are* (LI/LD)" can also be said as "$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is a* (LI/LD) set."

"Equivalent" conditions for LI/LD.

Again "equivalent" means that if the vectors are LI in any one of the following descriptions, they are automatically LI in all the others, and same for LD.

(**1**) $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a (LI/LD) set if (none of/at least one of) the vectors is in the span of the others. (Note LD can mean $\mathbf{v}_1 \in \operatorname{span}\{\mathbf{v}_2, \mathbf{v}_7\}$, it could be $\mathbf{v}_3 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$, or any other set of "span-ers"; all that is required is one case of TRUE.)

(**2**) Using $\mathbf{x} = (x_1, \ldots, x_k)$ (remember tuples are equivalent to col. vects.), if

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}_{n\times 1}, \qquad \text{AKA} \qquad \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_k\end{bmatrix}\begin{bmatrix}x_1\\ \vdots \\ x_k\end{bmatrix} = \mathbf{0}_{n\times 1}, \qquad \text{AKA} \qquad \mathbf{A}\mathbf{x} = \mathbf{0}_{n\times 1}$$

has (only the *trivial solution* $\mathbf{x} = \mathbf{0}_{k\times 1}$ / some *nontrivial solutions* $\mathbf{x} \neq \mathbf{0}_{k\times 1}$), then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are (LI/LD).

(**3**) If we write $V = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, then, by definition, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a *spanning set* for V. In this case then, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ (being/not being) a basis for V is the same as $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ being (LI/LD).

(**4**) If the determinant* $\det(\begin{bmatrix}\mathbf{v}_1 & \ldots & \mathbf{v}_k\end{bmatrix})$ is ($= 0$/$\neq 0$) then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is (LI/LD).

*Again we can only use this if $\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_k\end{bmatrix}$ is a square matrix.

Subspaces

Definition. If V is a *vector space* (so $\mathbb{R}^2$, $\mathbb{R}^3$, etc.) and W is a *subset* of V (so W is a set of vectors from V), then W is a subspace of V if all three conditions are true:

(**0**) The *zero vector* $\mathbf{0}$ is contained in W. Note that the dimensions of $\mathbf{0}$ depend on what V is—if $V = \mathbb{R}^2$ then the zero vector $\mathbf{0}$ is $\mathbf{0}_{2\times 1}$, if $V = \mathbb{R}^3$ it is $\mathbf{0}_{3\times 1}$, etc..

(**CA**) ("Closed under addition") If $\mathbf{u}$ and $\mathbf{v}$ are any two vectors from W, then the sum $\mathbf{u} + \mathbf{v}$ is also in W.

(**CS**) ("Closed under scalar multiplication") If $\mathbf{u}$ is any vector from W, then any scalar multiple $k\mathbf{u}$ is also in W.

Notes:

- The set $\{\mathbf{0}\}$ is a subspace of V, because of course combining $\mathbf{0}$ with itself doesn't leave the set $\{\mathbf{0}\}$.
- The entire set V is a subspace of V, because of course adding vectors in, say $\mathbb{R}^3$, never leaves $\mathbb{R}^3$. [See the next item].
- It may sound odd to say that "V is a subspace of V", but this is technically correct according to the definition. In practice, the term "*proper subspace*" means a subspace of V that is not simply V itself. By counting V as a subspace, we avoid having to constantly say things like "this theorem is true for all subspaces of V ... *and for V itself too.*"

Theorem. *If W is a subset of a vector space V, and there are some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ (from V) such that $W = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, then W is automatically a subspace of V.*

Theorem ("vice versa")**.** *If W is a subspace of a vector space V, then it is always possible to find a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ from V such that $W = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.*

- Note that there can be (infinitely) many sets of vectors that all span a subspace W.

Null space

Definition (null($\mathbf{A}$))**.** If $\mathbf{A}$ is an $(n \times k)$ matrix, then the null space of $\mathbf{A}$ is

$$\text{null}(\mathbf{A}) = \left\{\mathbf{x} \in \mathbb{R}^k : \mathbf{A}\mathbf{x} = \mathbf{0}_{n\times 1}\right\}.$$

- The null space null($\mathbf{A}$) is simply the set of solutions for the system/vector-equation $\mathbf{A}\mathbf{x} = \mathbf{0}$.
- null($\mathbf{A}$) always contains $\mathbf{0}$. If null($\mathbf{A}$) $= \{\mathbf{0}\}$, meaning null($\mathbf{A}$) contains *only* $\mathbf{0}$, then the columns of $\mathbf{A}$ are (LI). If null($\mathbf{A}$) contains the $\mathbf{0}$ *and more*, then the columns of $\mathbf{A}$ are LD.

Bases and dimension

Definition. If V is a vector space and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a set of vectors from V, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is said to be a basis for V if both:

(1) The linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ "generates/covers" V. Precisely, if

$$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = V.$$

(2) The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an LI set.

Definition ("Dimension")**.** The dimension of a vector space V (or some subspace W within V) is the **exact** number of *linearly independent* vectors **required** to make a basis of V (or W). This is denoted $\dim(V)$.

- Naturally $\dim(\mathbb{R}^n) = n$.
- When you start with some new/random V or W, it may not be clear what the dimension is at first; however, once you are able to find an example of a basis, the dimension must be the length of that list (basis).

Theorem. *The dimension $\dim(V)$ is "unique", in that any two bases of V must have the same number of (lin. indep.) vectors.*

Say V is a vector space and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a set of k vectors from V.

(1) If $k < \dim(V)$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ cannot possibly be a spanning set for V. (It may or may not be LI).

(2) If $k > \dim(V)$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ cannot be LD. (It may or may not be a spanning set for V).

(3) If $k = \dim(V)$, then it is not *a-priori* clear if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is/is-not a spanning set or is LI/LD—you simply have to check using one of our previous methods.

Row/Column space and Rank

Definition. If $\mathbf{A}$ has columns $\mathbf{c}_1, \ldots, \mathbf{c}_k$, so $\mathbf{A} = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_k \end{bmatrix}$, the column space of $\mathbf{A}$ is

$$\text{col}(\mathbf{A}) = \text{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_k\}.$$

- $\text{col}(\mathbf{A})$ is basically what vectors can be made as combinations of the column vectors that make up $\mathbf{A}$.

How to compute a basis for $\text{col}(\mathbf{A})$**:**

(1) Starting with $\mathbf{A}$, reduce $\mathbf{A}$ to echelon form (EF)*. [*Here "plain" EF is fine].

(2) Locate the *pivot columns* of (EF of $\mathbf{A}$).

(3) The columns from the original $\mathbf{A}$ in those pivot columns make a basis for $\text{col}(\mathbf{A})$.

Definition. The column rank $\text{col-rank}(\mathbf{A}) \overset{\text{DEF}}{=} \dim(\text{col}(\mathbf{A}))$.

Row vectors and row space.

Row vectors work the same as column vectors, just with a different shape. Our definitions and vocabulary all work equally well with rows vs. columns, especially since the transpose of the row vector is a column vector, and vice versa. For instance, if $\mathbf{r}_1, \ldots, \mathbf{r}_n$ are row vectors (of the same size), we say they are LI if making

$$c_1\mathbf{r}_1 + \cdots + c_n\mathbf{r}_n = \mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$$

is only possible with the "*trivial solution*" $(c_1, \ldots, c_n) = (0, \ldots, 0)$. If some nontrivial combination works, then the row vectors are LD.

Definition. If $\mathbf{A}$ has rows $\mathbf{r}_1, \ldots, \mathbf{r}_n$, so that $\mathbf{A} = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_n \end{bmatrix}$, the row space of $\mathbf{A}$ is

$$\text{row}(\mathbf{A}) = \text{span}\{\mathbf{r}_1, \ldots, \mathbf{r}_n\}.$$

How to compute a basis for $\text{row}(\mathbf{A})$**:**

(1) Starting with $\mathbf{A}$, reduce $\mathbf{A}$ to echelon form (EF)*. [*Again "plain" EF is fine].

(2) Locate the *pivot rows* of (EF of $\mathbf{A}$).

(3) The row vectors from (EF of $\mathbf{A}$) (not from the original $\mathbf{A}$) make a basis for $\text{row}(A)$.

Definition. The row rank $\text{row-rank}(\mathbf{A}) \overset{\text{DEF}}{=} \dim(\text{row}(\mathbf{A}))$.

Theorem. *The column rank of a matrix* $\mathbf{A}$ *is always equal to the row rank of* $\mathbf{A}$*. That is,*

$$\text{col-rank}(\mathbf{A}) = \text{row-rank}(\mathbf{A}).$$

This is because both ranks are equal to the number of pivots in (EF of $\mathbf{A}$*).*

⋆ **Note/Definition:** For this reason we just write $\text{rank}(\mathbf{A})$ and call it "the" rank of $\mathbf{A}$.

Note: Row ops do not change the row space of a matrix, but they do change the column space. (OTOH, column ops would not change the column space, but *would* change the row space.)

Helpful extras

(1) We said that

$$\operatorname{rank}(\mathbf{A}) = \Big(\#\ \textit{pivot columns in}\ (\text{EF }\mathbf{A})\Big)$$
$$\dim(\operatorname{null}(\mathbf{A})) = \Big(\#\ \textit{“free” columns in}\ (\text{EF }\mathbf{A})\Big)$$

Naturally then,

$$\operatorname{rank}(\mathbf{A}) + \dim(\operatorname{null}(\mathbf{A})) = \Big(\textit{total}\ \#\ \textit{of columns of}\ \mathbf{A}\Big)$$
$$= \Big(\#\ \textit{of variables in eq.}\ \mathbf{A}\mathbf{x} = \mathbf{0}\Big).$$

This is sometimes called the “rank-nullity theorem”, but we don’t need the name.

(2) (Null space as “redundancy”) Given a target vector $\mathbf{b}$, suppose that you have found a solution $\mathbf{v}$ for the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ (so that $\mathbf{A}\mathbf{v} = \mathbf{b}$). Then, if $\mathbf{w}$ is any vector from $\operatorname{null}(\mathbf{A})$, we can build a “new” solution $\tilde{\mathbf{v}} = \mathbf{v} + \mathbf{w}$, since

$$\mathbf{A}\tilde{\mathbf{v}} = \mathbf{A}(\mathbf{v} + \mathbf{w}) = \mathbf{A}\mathbf{v} + \mathbf{A}\mathbf{w} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

I said “new” because if $\mathbf{w} = \mathbf{0}$, then $\tilde{\mathbf{v}} = \mathbf{v}$ is not actually a different solution.

For the same reasons as above, $\tilde{\tilde{\mathbf{v}}} = \mathbf{v} + 2\mathbf{w}$ would *also* be a solution for $\mathbf{A}\mathbf{x} = \mathbf{b}$ (unless $\mathbf{w} = \mathbf{0}$), and we can use $\mathbf{w}$ this way to make infinitely many solutions to $\mathbf{A}\mathbf{x} = \mathbf{b}$. This is why $\operatorname{null}(\mathbf{A})$ essentially tells you how many different/redundant solutions you have* for an equation $\mathbf{A}\mathbf{x} = \mathbf{b}$

(*assuming that you found at least one solution $\mathbf{v}$ to begin with—if the equation has *no* solutions then there isn’t anything to discuss.)

LI conditions

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors from $\mathbb{R}^n$ (so they are $n \times 1$ vectors), and let

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{bmatrix} \qquad (\text{so } \mathbf{A} \text{ is } n \times k).$$

The following are all “equivalent” to $\mathbf{v}_1, \ldots, \mathbf{v}_k$ being LI. Meaning, if one is true then all are true, and if one is false then all are false.

(1) The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is LI (or “the columns of $\mathbf{A}$ are LI.”)

(2) No one of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is in the span of any (number of-) the others.

(3) Using $\mathbf{x} = (x_1, \ldots, x_k)$ (remember tuples are equivalent to col. vects.), the equation

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}_{n\times 1}, \qquad \text{AKA} \qquad \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \mathbf{0}_{n\times 1}, \qquad \text{AKA} \qquad \mathbf{A}\mathbf{x} = \mathbf{0}_{n\times 1}$$

has only the *trivial solution* $\mathbf{x} = \mathbf{0}_{k\times 1}$ as a solution. ($\operatorname{null}(\mathbf{A}) = \{\mathbf{0}_{k\times 1}\}$)

(4) Given any “target” vector $\mathbf{b} \in \mathbb{R}^n$, the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ either has *no* solution, or has a *unique* solution.

(5) The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ form a *basis* for $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, AKA $\operatorname{col}(\mathbf{A})$.

(6) $\dim(\operatorname{col}(\mathbf{A})) = k$ ($\operatorname{rank}(\mathbf{A}) = k$)

These additional conditions only apply if $\mathbf{A}$ is a *square* matrix.

(7) The determinant $|\mathbf{A}| \neq 0$.

(8) The matrix $\mathbf{A}$ is *invertible* (or "$\mathbf{A}$ is nonsingular")

(9) The (REF of $\mathbf{A}$) is equal to $\mathbf{I}_n$. (or "$\mathbf{A}$ is row-equivalent to $\mathbf{I}_n$.")

(10) Given any "target" vector $\mathbf{b} \in \mathbb{R}^n$, the equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

has *unique* solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

LD CONDITIONS

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors from $\mathbb{R}^n$ (so they are $n \times 1$ vectors), and let

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{bmatrix} \qquad \text{(so } \mathbf{A} \text{ is } n \times k\text{)}.$$

The following are all "equivalent" to $\mathbf{v}_1, \ldots, \mathbf{v}_k$ being LD. Meaning, if one is true then all are true, and if one is false then all are false.

(1) The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is LD (or "the columns of $\mathbf{A}$ are LD.")

(2) At least one of the $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is in the span of the others. (Note this can mean $\mathbf{v}_1 \in \operatorname{span}\{\mathbf{v}_2, \mathbf{v}_7\}$, it could be $\mathbf{v}_3 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$, or any other set of "span-ers"; all that is required is one case of TRUE.)

(3) Using $\mathbf{x} = (x_1, \ldots, x_k)$ (remember tuples are equivalent to col. vects.), the equation

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}_{n\times 1}, \qquad \text{AKA} \qquad \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \mathbf{0}_{n\times 1}, \qquad \text{AKA} \qquad \mathbf{A}\mathbf{x} = \mathbf{0}_{n\times 1}$$

has some *nontrivial solutions* $\mathbf{x} \neq \mathbf{0}_{k\times 1}$ as a solution. (So null($\mathbf{A}$) contains more than just $\mathbf{0}_{k\times 1}$.)

(4) Given any "target" vector $\mathbf{b} \in \mathbb{R}^n$, the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has either *no* solution, or has *infinitely many* solutions (so unique solution cannot happen).

(5) The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *do not* form a basis for $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, AKA col($\mathbf{A}$). (Some subset of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ will make a basis though.)

(6) $\dim(\operatorname{col}(\mathbf{A})) < k$ (so $\operatorname{rank}(\mathbf{A}) < k$)

These additional conditions only apply if $\mathbf{A}$ is a *square* matrix.

(7) The determinant $|\mathbf{A}| = 0$.

(8) The matrix $\mathbf{A}$ is *noninvertible* (or "$\mathbf{A}$ is singular")

(9) The (REF of $\mathbf{A}$) is NOT equal to $\mathbf{I}_n$. (or "$\mathbf{A}$ is NOT row-equivalent to $\mathbf{I}_n$.")