Subspaces of $\mathbb{R}^n$ (in $\mathbb{R}^n$ we use vectors $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$)

Let W be some subset of $\mathbb{R}^n$, so W is a collection of (column) vectors from $\mathbb{R}^n$.

Def W is a subspace of $\mathbb{R}^n$ if 3 things are true:

(0) $\underline{0}$-vector is contained in W.

(CA) ("closed under addition") If $\underline{u}$ and $\underline{v}$ are any two vectors from W, then $\underline{u}+\underline{v}$ should also be in W ($\underline{u}, \underline{v} \in W \Rightarrow \underline{u}+\underline{v} \in W$ too)

(CS) ("closed under scaling") If $\underline{u}$ is any vector from W, then any scalar multiple $c\underline{u}$ is also in W. (succinctly, $\underline{u} \in W \Rightarrow c\underline{u} \in W$)

Ex: $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 3x - 2y - z = 0 \right\}$

$z = 3x - 2y$

Method 1 (from definitions)

(0) is $\underline{0}$ in W? Yes, because coordinates of $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ satisfy $3(0) - 2(0) - 0 = 0$ ✓

(CA) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 3x - 2y - z = 0 \right\}$

$= \left\{ \begin{bmatrix} x \\ y \\ 3x-2y \end{bmatrix} : x, y \in \mathbb{R} \right\}$

$\underline{u} = \begin{bmatrix} a \\ b \\ 3a-2b \end{bmatrix}$, $\underline{v} = \begin{bmatrix} c \\ d \\ 3c-2d \end{bmatrix}$

$\underline{u}+\underline{v} = \begin{bmatrix} a+c \\ b+d \\ 3(a+c) - 2(b+d) \end{bmatrix}$. This fits into the description of W.

so W closed under addition.

CS Take $\underline{u} \in W$ $\underline{u} = \begin{bmatrix} x \\ y \\ 3x-2y \end{bmatrix}$

will $k\underline{u} \in W$ too? $k\underline{u} = \begin{bmatrix} kx \\ ky \\ k(3x-2y) \end{bmatrix} = \begin{bmatrix} kx \\ ky \\ 3(kx)-2(ky) \end{bmatrix}$

so any multiple $k\underline{u}$ will be in W, and so W is "closed under scalar multiplication."

So $\mathbf{0}$, CA, CS all true, so W is a subspace of $\mathbb{R}^3$

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 3x-2y-z=0 \right\}$$

<u>Method: write W as a span:</u>

We ~~to~~ rewrote $W = \left\{ \begin{bmatrix} x \\ y \\ 3x-2y \end{bmatrix} : \overbrace{x, y \in \mathbb{R}}^{x,y \text{ are free/unrestricted}} \right\}$

$$\begin{bmatrix} x \\ y \\ 3x-2y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \quad x, y \text{ free.}$$

so $W = \left\{ a\underbrace{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}_{\underline{u}} + b\underbrace{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}_{\underline{v}} : a, b \in \mathbb{R} \right\}$

$W = \underline{\text{span}\{\underline{u}, \underline{v}\}} = \{ a\underline{u} + b\underline{v} : a, b \in \mathbb{R} \}.$

✓ $\mathbf{0}$ $\mathbf{0} \in W$? Yes, because $a=b=0 \implies 0\underline{u} + 0\underline{v} = \underline{\mathbf{0}}$

✓ CA If we take vectors $\underline{X} = a\underline{u} + b\underline{v} \in W$
$\underline{Y} = c\underline{u} + d\underline{v} \in W$

$\underline{X} + \underline{Y} \in W$? Yes, because $\underline{X} + \underline{Y}$
$= (a+c)\underline{u} + (b+d)\underline{v}$

✓ CS If $\underline{X} = a\underline{u} + b\underline{v}$, is $k\underline{X} \in W$?

Yes, $k\underline{X} = (ka)\underline{u} + (kb)\underline{v} \in W$

So, W is a subspace of $\mathbb{R}^3$.

"**Thm**" If W is a subset of $\mathbb{R}^n$, and you write W as a span of some vectors from $\mathbb{R}^n$

(Ex: $W = \text{span}\{\underline{u}\}$ or $W = \text{span}\{\underline{u}, \underline{v}\}$, or $W = \text{span}\{\underline{u}, \underline{v}, \underline{w}\}$ . . .)

then automatically W is a subspace of $\mathbb{R}^n$.

Def: (Null space): $\underline{A}$ $(m \times n)$, null$(\underline{A})$ is the set of solutions to vector equation $\underline{A}\underline{x} = \underline{0}$

$$\text{null}(\underline{A}) = \{ \underline{x} : \underline{A}\underline{x} = \underline{0} \}$$

$(3\times4)(4\times1) \to 3\times1$

Ex:
$$\underbrace{\begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix}}_{\underline{A}\ (3\times4)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\underline{x}\ (4\times1)} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{0}}$$

$$\text{null}(\underline{A}) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : \underline{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underline{0}_{(3\times1)} \right\}$$

is null$(\underline{A})$ a subspace of $\mathbb{R}^4$? How do we "generate" it?

Compute solutions $\underline{x}$ using REF

augmented matrix
$$\left[\begin{array}{cccc|c} 1 & 3 & -15 & 7 & 0 \\ 1 & 4 & -19 & 10 & 0 \\ 2 & 5 & -26 & 11 & 0 \end{array}\right]$$

. . . to REF . . .
$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & -3 & -2 & 0 \\ 0 & \boxed{1} & -4 & 3 & 0 \\ \underbrace{0 \quad 0}_{\substack{x_1, x_2 \\ \text{pivot}}} & & \underbrace{0 \quad 0}_{\substack{x_3, x_4 \\ \text{free}}} & & 0 \end{array}\right]$$

$x_3 = s$
$x_4 = t$ free params.

Solutions to $\underline{A}x = \emptyset$ are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s+2t \\ 4s-3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

s, t free.

So solutions to $\underline{A}x = \emptyset$, aka null($\underline{A}$), are

$$\text{null}(\underline{A}) = \left\{ s\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$ So null($\underline{A}$) is a span of some

~~#~~ vectors, it is a subspace of $\mathbb{R}^4$.

Could also check if null($\underline{A}$) is a subspace directly:

$$\text{null}(\underline{A}) = \{ x \in \mathbb{R}^4 : \underline{A}x = \emptyset \}$$ $\underline{A}: (3\times4)$

$x: (4\times1)$

✓ ⓪ does $\underline{A}\,\emptyset_{4\times1} = \emptyset_{3\times1}$? Yes.

✓ (CA) If $\underline{A}x = \emptyset$ and $\underline{A}y = \emptyset$, is $x+y \in \text{null}(\underline{A})$?

Yes, b/c $\underline{A}(x+y) = \underline{A}x + \underline{A}y = \emptyset + \emptyset = \emptyset$.

✓ (CS) If $\underline{A}x = \emptyset$, does $\underline{A}(kx) = \emptyset$? Yes, because

$$\underline{A}(kx) = k\underline{A}x = k\emptyset = \emptyset.$$

null($\underline{A}$) is a subspace of $\mathbb{R}^4$.