Systems of 1st order linear ODEs

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Eigenvalues and eigenvectors

Definition. Let $\mathbf{A}$ be given *square* matrix. If $\mathbf{v}$ is a nonzero vector such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \qquad \text{for some constant } \lambda,$$

then we say that "$\mathbf{v}$ is an eigenvector for $\mathbf{A}$ with eigenvalue λ." This description is identical to saying that

$$\mathbf{v} \in \text{null}(\mathbf{A} - \lambda\mathbf{I}).$$

Computing eigenvalues and eigenvectors:

(1) The eigenvalues of a matrix $\mathbf{A}$ are the λ (which can be real or complex) that are roots of the characteristic polynomial

$$\det(\mathbf{A} - \lambda\mathbf{I}).$$

(2) For each λ that is an eigenvalue, to find its associated eigenvectors, we simply find the solutions to the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$. That is, we compute/determine

$$E_\lambda \overset{\text{DEF}}{=} \text{null}(\mathbf{A} - \lambda\mathbf{I}),$$

and then pick out a *basis* for that set to serve as our "representative" eigenvectors.

Notes and vocabulary.

- Eigenvalues and -vectors depend on the given matrix $\mathbf{A}$.
- For any eigenvalue λ, the space of eigenvectors E_λ is a **subspace** of $\mathbb{R}^n$ (whatever $\mathbb{R}^n$ corresponds to the size of $\mathbf{A}$).
  - We know this automatically because $E_\lambda = \text{null}(\mathbf{A} - \lambda\mathbf{I})$, and nullspaces are subspaces.
  - We can also verify the subspace tests easily: (i) Any nullspace certainly contains $\mathbf{0}$; (ii) if

$$(\mathbf{A}\mathbf{v} = \lambda\mathbf{v}) \ \&\ (\mathbf{A}\mathbf{w} = \lambda\mathbf{w}) \quad \Longrightarrow \quad \mathbf{A}(\mathbf{v}+\mathbf{w}) = \mathbf{A}\mathbf{v} + \mathbf{A}\mathbf{w} = \lambda(\mathbf{v}+\mathbf{w}).$$

    (iii) Similarly, $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ implies that $\mathbf{A}(k\mathbf{v}) = k(\mathbf{A}\mathbf{v}) = \lambda(k\mathbf{v})$.
- algebraic multiplicity (of λ): The number of times λ occurs as a root of the char. poly.
- geometric multiplicity (of λ): The dimension $\dim E_\lambda$, AKA $\dim(\text{null}(\mathbf{A} - \lambda\mathbf{I}))$.

Systems of 1st-order linear ODEs

Idea: These are collections of 1st-order diff. eqs. involving multiple functions (ex: $x_1(t)$, $x_2(t)$, $x_3(t)$), where the functions have derivatives that are "interlinked" (in practice the term "coupled" is used) with the (1st-powers of-) the functions.

The system can be written in **scalar form** or **vector form**.

Example 1:

$$\begin{cases} x_1' = 0x_1 + 3x_2 - x_3 \\ x_2' = 1x_1 + 1x_2 + 2x_3 \\ x_3' = -2x_1 + 3x_2 + 0x_3 \end{cases} \qquad \text{(scalar form)}$$

is the same as

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{(vector form).}$$

Example 2:

$$\begin{cases} x_1' = 0x_1 + 3x_2 - x_3 + t^2 \\ x_2' = 1x_1 + 1x_2 + 2x_3 + \sin(t) \\ x_3' = -2x_1 + 3x_2 + 0x_3 \end{cases} \qquad \text{(scalar form)}$$

is the same as

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 3 & -1 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} t^2 \\ \sin(t) \\ 0 \end{bmatrix} \qquad \text{(vector form).}$$

Notes:

(1) The general (vector) form of a system of 1st order linear ODEs is

$$\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t), \qquad \mathbf{A} \text{ is } (n \times n), \qquad \mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \qquad \mathbf{F}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

The extra vector-valued function $\mathbf{F}(t)$ holds all the "non-x_i parts" of the equations.

(2) If we just have $\mathbf{X}' = \mathbf{A}\mathbf{X}$, meaning $\mathbf{F}(t) = \mathbf{0}$, then the system is **homogeneous**. Thus, Example 1 above is homogeneous, while Example 2 is not.

(3) **Caution:** The matrix $\mathbf{A}$ doesn't *have to* have only constant entries—it is fine if the coefficients of the x_i vary with t (see [L31/32] Notes for example).

Solutions to systems: If we have a system of n ODEs, individual solutions can be expressed either as a tuple/list

$$x_1(t) = (\cdots), \quad x_2(t) = (\cdots), \quad \cdots \quad x_n(t) = (\cdots), \qquad (\text{"}\textbf{scalar form}\text{"})$$

or (equivalently) as a vector

$$\mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} (\cdots) \\ (\cdots) \\ \vdots \\ (\cdots) \end{bmatrix} \qquad (\text{“}\textbf{vector form}\text{”}).$$

Theorem. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. The solutions of a homogeneous linear system* $\mathbf{X}' = \mathbf{A}\mathbf{X}$ *make an* n*-dimensional vector space: that is, describing the set*

$$\Big\{ \mathbf{X} : \mathbf{X}' = \mathbf{A}\mathbf{X} \Big\}$$

requires n *linearly independent solutions* $\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$*. With those* n *“basis solutions”, the general solution is*

$$\mathbf{X} = c_1 \mathbf{X}_1 + \cdots + c_n \mathbf{X}_n.$$

The Wronskian for vector-valued functions.

Definition. Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$, be vector-valued functions, with values in $\mathbb{R}^n$, and label the individual components of the $\mathbf{X}_j$ in the manner

$$\mathbf{X}_1(t) = \begin{bmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \quad \mathbf{X}_2(t) = \begin{bmatrix} x_{12}(t) \\ \vdots \\ x_{n2}(t) \end{bmatrix}, \quad \ldots \quad \mathbf{X}_n(t) = \begin{bmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}.$$

The Wronskian $W(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ is defined via

$$W(\mathbf{X}_1, \ldots, \mathbf{X}_n) = \det\big[\mathbf{X}_1 \quad \cdots \quad \mathbf{X}_n\big] = \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}$$

The eigenvalue method for homogeneous systems

In solving a homogeneous system

$$\mathbf{X}' = \mathbf{A}\mathbf{X}, \qquad \text{with} \qquad \mathbf{A}\ (n \times n), \quad \text{and} \quad \mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix},$$

we require n linearly independent solutions $\mathbf{X}_i(t)$. The eigenvalue method consists of three “definite” steps, and a fourth extra step if there are defects.

(1) Find the eigenvalues λ of $\mathbf{A}$ using the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I})$.

(2) Find/compute <u>linearly indep.</u> eigenvectors for the different eigenvalues λ.

(3) Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are the linearly independent eigenvectors* from Step 2, and that they have eigenvalues $\lambda_1, \ldots, \lambda_k$ (possible repeats). Then we make basis solutions

$$\mathbf{X}_1(t) = e^{\lambda t}\mathbf{v}_1, \qquad \ldots, \qquad \mathbf{X}_k(t) = e^{\lambda t}\mathbf{v}_k.$$

If indeed this produces n independent solutions (i.e., if $k = n$), then we are done.

(4) If we do **not** get n lin. indep. eigenvectors, and instead get *less than* n, then we need to deal with "generalized eigenvectors".

Solutions with complex eigenvalues/vectors. Suppose that $\mathbf{A}$ has a complex-conjugate pair of eigenvalues $\lambda = p \pm qi$.

- If $\mathbf{v}$ is an eigenvector for $\lambda = p + qi$, then automatically the conjugate $\bar{\mathbf{v}}$ will be an eigenvector for $\bar{\lambda} = p - qi$.
- Given the eigenvector $\mathbf{v}$, break it into real and imaginary parts:

$$\mathbf{v} = \mathbf{a} + \mathbf{b}i.$$

For example

$$\mathbf{v} = \begin{bmatrix} 3+4i \\ 1-2i \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + i\begin{bmatrix} 4 \\ -2 \end{bmatrix} \qquad \Longrightarrow \qquad \mathbf{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

- Instead of using basis solutions $\{e^{\lambda t}\mathbf{v}, e^{\bar{\lambda} t}\bar{\mathbf{v}}\}$, we make two alternate basis solutions by breaking $e^{\lambda t}\mathbf{v}$ into its real and imaginary parts:

$$\begin{aligned} e^{(p+qi)t}\mathbf{v} &= \big(\cos(qt) + i\sin(qt)\big)\big(\mathbf{a} + \mathbf{b}i\big) \\ &= \big(\mathbf{a}\cos(qt) - \mathbf{b}\sin(qt)\big) + i\big(\mathbf{b}\cos(qt) + \mathbf{a}\sin(qt)\big). \end{aligned}$$

That is, we find that our two alternate solutions are

$$\begin{aligned} \mathbf{X}_1 &= \mathrm{Re}(e^{\lambda t}\mathbf{v}) = \mathbf{a}\cos(qt) - \mathbf{b}\sin(qt), \\ \mathbf{X}_2 &= \mathrm{Im}(e^{\lambda t}\mathbf{v}) = \mathbf{b}\cos(qt) + \mathbf{a}\sin(qt). \end{aligned}$$

Dealing with generalized eigenvectors.

Definition. A vector $\mathbf{v}$ is a "genuine" eigenvector for $\mathbf{A}$, with eigenvalue λ, if

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}.$$

If we instead have

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} \neq \mathbf{0} \qquad \text{but} \qquad (\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v} = \mathbf{0},$$

then $\mathbf{v}$ is a (2-step) generalized eigenvector. If

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} \neq \mathbf{0}, \qquad (\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v} \neq \mathbf{0}, \qquad \text{but} \qquad (\mathbf{A} - \lambda\mathbf{I})^3\mathbf{v} = \mathbf{0},$$

then $\mathbf{v}$ is a (3-step) generalized eigenvector, and so forth.

*Note that k must be $\leq n$, and it is possible that we find less than n lin. indep. eigenvectors.

Definition (multiplicity/defect). Let $\mathbf{A}$ be a matrix and let λ be an eigenvalue of $\mathbf{A}$. Thus, λ is a root of the characteristic polynomial $|\mathbf{A} - \lambda\mathbf{I}|$.

(1) The <u>algebraic multiplicity</u> of λ is the number of times it occurs as a root of $|\mathbf{A} - \lambda\mathbf{I}|$.
(2) The <u>geometric multiplicity</u> of λ is $\dim(E_\lambda) = \dim(\mathrm{null}(\mathbf{A} - \lambda\mathbf{I}))$.
(3) The <u>defect</u> of λ is $d =$ (alg. mult. of λ) $-$ (geom. mult. of λ).
(4) If λ has defect 0, we can say that λ "is <u>complete</u>", or that "λ has a <u>complete set</u> of eigenvectors".

Solutions with generalized eigenvectors (defect = 1).

In solving $\mathbf{X}' = \mathbf{A}\mathbf{X}$, suppose that the eigenvalue λ occurs twice (has alg. mult. 2) and has defect $d = 1$. Then we infer that (geom. mult. of λ) is (defect d) $-$ (alg. mult.) $= 1$. Thus

$$\dim E_\lambda = 1, \qquad \text{and we pick out genuine eigenvector} \quad \mathbf{v}_1.$$

Then our two basis solutions are

$$\begin{aligned} \mathbf{X}_1 &= e^{\lambda t}\mathbf{v}_1, & (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 &= \mathbf{0}, \\ \mathbf{X}_2 &= e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2), & (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 &= \mathbf{v}_1. \end{aligned}$$

Solutions with generalized eigenvectors (defect $\geq$ 2).

If λ occurs 3 times, and we have defect $d = 2$, with $\mathbf{v}_1$ a genuine eig.vect., then we need to make a "3-step chain" of eig.vects.. Namely, we need $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ with

$$\begin{cases} (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0} & (\mathbf{v}_1 \text{ is genuine eig.vect.}), \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1 & (\mathbf{v}_2 \text{ is 2-step eig.vect.}), \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3 = \mathbf{v}_2 & (\mathbf{v}_3 \text{ is 3-step eig.vect.}). \end{cases}$$

There are two methods for making this "3-chain".

<u>Method 1:</u>

(1) Solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ to pick $\mathbf{v}_1 \neq \mathbf{0}$ (we assume this is already done).
(2) Solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{v}_1$ to find/pick $\mathbf{v}_2$.
(3) Solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{v}_2$ to find/pick $\mathbf{v}_3$.

<u>Method 2 (Preferred/Easier):</u>

(1) Find a $\mathbf{v} \neq \mathbf{0}$ such that $(\mathbf{A} - \lambda\mathbf{I})^3\mathbf{v} = \mathbf{0}$, and set that as "$\mathbf{v}_3$".
(2) Check that both $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3 \neq \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_3 \neq \mathbf{0}$. If one of those is $\mathbf{0}$, restart with a different $\mathbf{v}_3$.
(3) If neither is $\mathbf{0}$, then (re)set

$$\mathbf{v}_2 \overset{\text{DEF}}{=} (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3, \qquad \mathbf{v}_1 \overset{\text{DEF}}{=} (\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v}_3 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2.$$

Then these $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ make the desired "3-chain".

Once we have the desired "3-chain" with $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, our basis solutions are

$$\begin{aligned}
\mathbf{X}_1 &= e^{\lambda t}\mathbf{v}_1, && (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_1 = \mathbf{0},\\
\mathbf{X}_2 &= e^{\lambda t}\big(t\mathbf{v}_1 + \mathbf{v}_2\big), && (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1,\\
\mathbf{X}_3 &= e^{\lambda t}\big(\tfrac{1}{2}t^2\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3\big), && (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_3 = \mathbf{v}_2.
\end{aligned}$$

Remark. Using Method 2 above, it is easy to see how we could extend the method to handle making a "4-chain", "5-chain", etc.. For instance, with a 4-chain we would add $\mathbf{v}_4$ and another basis solution

$$\mathbf{X}_4 = e^{\lambda t}\big(\tfrac{1}{6}t^3\mathbf{v}_1 + \tfrac{1}{2}t^2\mathbf{v}_2 + t\mathbf{v}_3 + \mathbf{v}_4\big), \qquad (\mathbf{A}-\lambda\mathbf{I})\mathbf{v}_4 = \mathbf{v}_3.$$

Phase plane portraits

Eigenvalues λ_1, λ_2 are both real.

(1) Eigenvalue $\lambda = 0$ occurs:
   - (a) $\lambda = 0$ occurs once, eig.vect. $\mathbf{v}$: **parallel lines** *perpendicular* to the $\mathbf{v}$-line.
   - (b) $\lambda = 0$ occurs twice, defect $d = 1$: **parallel lines** *parallel* to the $\mathbf{v}$-line.
   - (c) $\lambda = 0$ occurs twice, defect $d = 0$: **all points staionary**.

(2) Distinct eigenvalues $\lambda_1 \neq \lambda_2$, both $\neq 0$:
   - (a) Different sign: **saddle point**.
   - (b) Same sign: **improper nodal source/sink**.

(3) Single eigenvalue $\lambda \neq 0$ repeated:
   - (a) Defect $d = 0$: **proper nodal source/sink**.
   - (b) Defect $d = 1$, w/ genuine eig.vect. $\mathbf{v}_1$: **improper nodal source/sink**; curves "begin" parallel to $\pm\mathbf{v}_1$, and U-turn to "end" parallel to $\mp\mathbf{v}_1$.

Eigenvalues λ_1, λ_2 are a complex-conjugate pair.

Say the conjugate pair is

$$\lambda = p \pm qi, \qquad \text{so} \qquad \operatorname{Re}\lambda = p, \quad \operatorname{Im}\lambda = q.$$

(1) $\operatorname{Re}\lambda > 0$: **spiral source**

(2) $\operatorname{Re}\lambda = 0$: **center**

(3) $\operatorname{Re}\lambda < 0$: **spiral sink**.

Note: To determine the direction of rotation (CW/CCW), pick a position $\mathbf{x}$ (e.g. $\mathbf{x} = (1,0)$) and compute the "velocity vector" $\mathbf{x}' = \mathbf{A}\mathbf{x}$ at that position, and draw the vector to see where it points.