Practice determining what kind of (substitution) equation.

Recall that we learned 3 types of substitution equations:

1) Equations that can be expressed as

$$F(ax+by+c),$$ (Here "$F(\cdot)$" is just some expression)

for which we used $v = ax+by+c$;

2) Equations that can be written as

$$F(y/x),$$

for which we use the substitution $v = y/x$;

These are "<u>homogeneous</u>" equations; see the old note we had on types of substitution

3) Equations of the form

$$y' + P(x)\,y = Q(x)\,y^n$$

are <u>Bernoulli equations.</u>

Some further remarks

1) <u>Equations can fall into multiple categories of the above.</u> For example, though <u>linear</u> eqs aren't listed above, since we didn't solve them via substitution, we see that an equation like

$$y' + \frac{2x}{x^2+1}\,y = 2x$$

is a linear equation, and is a Bernoulli equation (with $n=0$, so that $y^n = 1$)

2) Similarly, it is quite possible to have an equation that is both Bernoulli and homogeneous. For example:

$$y' - \frac{3}{x} y = \frac{y^2}{x^2}$$

On the one hand, we see that this is

$$y' + P(x) = Q(x) y^2 \quad \begin{pmatrix} P(x) = -\frac{3}{x} \\ Q(x) = \frac{1}{x^2} \end{pmatrix},$$

So it is Bernoulli $(n=2)$. On the other hand, we have

$$y' = 3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2,$$

$$:= F(y/x),$$

where $F(\cdot) = 3(\cdot) + (\cdot)^2$,

so this is also a homogeneous equation.

(in this case, either substitution $v = y^{1-n} = y^{-1}$ or $v = y/x$ will work and get you the same answer).

3) Remember (from the previous note) that you can check if an equation is homogeneous by checking to see if the degrees of all terms match on both sides (remember that we ignore $\frac{dy}{dx}$ when checking this, since it has degree 0)

1.6 Problems

Find general solutions of the differential equations in Problems 1 through 30. Primes denote derivatives with respect to x throughout. 45 47 49

1. $(x+y)y' = x - y$ 2. $2xyy' = x^2 + 2y^2$ 51
3. $xy' = y + 2\sqrt{xy}$ 4. $(x-y)y' = x + y$ 53
5. $x(x+y)y' = y(x-y)$ 6. $(x+2y)y' = y$ 55
7. $xy^2y' = x^3 + y^3$ 8. $x^2y' = xy + x^2e^{y/x}$
9. $x^2y' = xy + y^2$ 10. $xyy' = x^2 + 3y^2$ 56
11. $(x^2 - y^2)y' = 2xy$
12. $xyy' = y^2 + x\sqrt{4x^2 + y^2}$
13. $xy' = y + \sqrt{x^2 + y^2}$
14. $yy' + x = \sqrt{x^2 + y^2}$
15. $x(x+y)y' + y(3x+y) = 0$
16. $y' = \sqrt{x+y+1}$ 17. $y' = (4x+y)^2$ 57
18. $(x+y)y' = 1$ 19. $x^2y' + 2xy = 5y^3$
20. $y^2y' + 2xy^3 = 6x$ 21. $y' = y + y^3$
22. $x^2y' + 2xy = 5y^4$ 23. $xy' + 6y = 3xy^{4/3}$ 58
24. $2xy' + y^3e^{-2x} = 2xy$
25. $y^2(xy' + y)(1 + x^4)^{1/2} = x$
26. $3y^2y' + y^3 = e^{-x}$ 59
27. $3xy^2y' = 3x^4 + y^3$
28. $xe^y y' = 2(e^y + x^3e^{2x})$
29. $(2x \sin y \cos y)y' = 4x^2 + \sin^2 y$
30. $(x + e^y)y' = xe^{-y} - 1$

Let's do a few practice problems setting up what kinds of substitutions / equations we know.

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1) $(x+y)y' = x - y$

$$y' = \frac{x-y}{x+y} = \frac{x(1 - y/x)}{x(1 + y/x)} = \frac{1 - (y/x)}{1 + (y/x)},$$

so this is <u>homogeneous</u>. It's not Bernoulli since we <u>can't</u> split the fraction to make

$$y' + P(x)y = Q(x)y^n$$

2) $$2xy\,y' = x^2 + 2y^2$$

$$y' = \frac{x^2+2y^2}{2xy} \begin{array}{l} \leftarrow \text{degree } 2 \\ \leftarrow \text{degree } 1+1=2 \end{array}$$

$$= \frac{x^2\left(1+2y^2/x^2\right)}{x^2\left(2y/x\right)}$$

$$= \frac{1+2(y/x)^2}{2(y/x)} = F(y/x),$$

so this is <u>homogeneous</u>. On the other hand

$$y' = \frac{x^2+2y^2}{2xy} = \frac{x}{2y} + \frac{y}{x} = \frac{1}{2}xy^{-1} + \frac{1}{x}y$$

$$\Longrightarrow \quad y' - \frac{1}{x}y = \frac{1}{2}x\,y^{-1},$$

so $y' + P(x)y = Q(x)y^n$ $\quad \left(\begin{array}{l} P(x) = -\frac{1}{x} \\ Q(x) = \frac{x}{2} \\ n = -1 \end{array} \right)$

so this is <u>Bernoulli</u> too. Again, either substitution $v = y/x$ or $v = y^{1-n} = y^2$ will work.

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3) $$xy' = \overset{\text{degree } 1}{y} + 2\sqrt{xy}.$$

$$\underbrace{y'}_{\text{degree } 1} = \frac{y}{x} + 2\frac{\sqrt{xy}}{x}$$

$\sqrt{xy} = x^{\frac{1}{2}}y^{\frac{1}{2}}$ is degree $\frac{1}{2}+\frac{1}{2}=1$.

$$= y/x + 2\sqrt{\frac{xy}{x^2}}$$

$$= y/x + 2\sqrt{y/x}$$

$$= F(y/x)$$

so it is <u>homogeneous</u>.

Is it <u>Bernoulli? Yes:</u>

$$y/x + 2\sqrt{y/x}$$

$$= x^{-1}\cdot y + 2x^{-1/2}y^{1/2},$$

so $y' = x^{-1}y + (2x^{-1/2})y^{1/2}$

$$y' \underbrace{- x^{-1}}_{P(x)} y = \underbrace{(2x^{-1/2})}_{Q(x)} y^{1/2}$$

$Q(x)\,y^n$; $n = 1/2$

6) $(x+2y)y' = y.$

$$y' = \frac{y^{\text{degree 1}}}{\underbrace{x+2y}_{\text{degree 1}}} = \frac{y}{x(1+2y/x)} = \frac{(y/x)}{1+2(y/x)},$$

So this is homogeneous. It's not Bernoulli because there's no way to split the fraction $\frac{y}{x+2y}$ to make $y' + P(x)y = Q(x)y^n$.

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20) $\underbrace{y^2}_{\text{degree 2}} y' + \underbrace{2xy^3}_{\text{degree 4}} = \underbrace{6x}_{\text{degree 1}}$

Degrees don't match, so it's <u>not</u> homogeneous.

$y' + 2xy^1 = 6xy^{-2}$ is clearly Bernoulli though.