Ex Find a general solution for ODE

$$(x^2+1)y' + 3xy = 6x$$

$y' + Py = Q \longrightarrow P(x) = \ldots$ careful ! Need to first put eq into standard form $y' + P(x)y = Q(x)$

$$\boxed{y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}} \longrightarrow y' + P\cdot y = Q(x)$$

$P(x) = \dfrac{3x}{x^2+1}$, $\quad \rho(x) = e^{\int P(x)\,dx} = e^{\frac{3}{2}\ln|x^2+1| + k}$

$Q(x) = \dfrac{6x}{x^2+1}$, $\quad = e^k |\underbrace{x^2+1}_{>0 \text{ always}}|^{3/2} = e^k (x^2+1)^{3/2}$

$$\rho y' + \rho' y = \rho Q \Longrightarrow \frac{d}{dx}[\rho \cdot y] = \rho Q$$

$$\longrightarrow \frac{d}{dx}\left(e^k (x^2+1)^{3/2} \cdot y\right) = e^k (x^2+1)^{3/2} \cdot \frac{6x}{x^2+1}$$

$$\Rightarrow \cancel{e^k}(x^2+1)^{\frac{3}{2}} y = \cancel{e^k} \int 6x (x^2+1)^{1/2}\,dx + \underbrace{Ce^{-k}}_{=C}$$

$$\Rightarrow (x^2+1)^{3/2} y = 2(x^2+1)^{\frac{3}{2}} + C$$

$$\Rightarrow \underline{y(x) = 2 + C(x^2+1)^{-3/2}}$$

☆ Notice how k didn't matter? Can always pick $k = 0 \Longrightarrow \rho(x) = (x^2+1)^{3/2}$

Ex Find a gen. sol. for $\{xy' - 9y = x^{10}\cos(x)\}$, assuming $x > 0$ *

Need $y' + P(x)y = Q(x)$

$$y' - \frac{9}{x}y = x^9\cos(x)$$

$P(x) = -9/x$

$Q(x) = x^9\cos(x)$

SET $\rho(x) = e^{\int P(x)\,dx} = e^{\int -9/x\,dx}$

$= e^{-9\ln|x| + k}$ (k picked $= 0$)

$= |x|^{-9} \longrightarrow x^{-9}$

(* b/c $x > 0$)

so $y' - \frac{9}{x}y = x^9\cos(x)$

$\Rightarrow x^{-9}y' - 9x^{-8}y = \cos(x)$

$\Rightarrow \int \frac{d}{dx}\left[x^{-9}y\right]dx = \int \cos(x)\,dx$

$\Rightarrow x^{-9}y = \sin(x) + C$

$$\underline{y(x) = x^9\sin(x) + Cx^9}$$