RESEARCH STATEMENT

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1 Introduction

The *partitions* of $n \in \mathbb{N}$ are tuples of positive integers $(a_1, a_2, \ldots, a_k)$ such that $a_1 \geq a_2 \geq \ldots \geq a_k$ and $a_1 + a_2 + \cdots + a_k = n$. Using $\Pi[n]$ to denote the set of such partitions, the quantities $\mathfrak{p}(n) := \mathrm{card}(\Pi[n])$ are the (ordinary) *partition numbers*. In 1918 Hardy and Ramanujan used the functional equation of the Dedekind η-function to give an asymptotic formula for $\mathfrak{p}(n)$ as $n \to \infty$; namely, in [11] they establish (in addition to stronger results) that

$$\log \mathfrak{p}(n) \sim \kappa\sqrt{n}, \qquad \text{where } \kappa := \pi\sqrt{2/3}, \tag{1.1}$$

and the relation $a(n) \sim b(n)$ indicates that $\lim_{n\to\infty} a(n)/b(n) = 1$. At nearly the same time Ramanujan announced and proved his eponymous "congruences", which are the relations that

$$\mathfrak{p}(5n+4) \equiv 0 \pmod 5, \quad \mathfrak{p}(7n+5) \equiv 0 \pmod 7, \quad \text{and} \quad \mathfrak{p}(11n+6) \equiv 0 \pmod{11}.$$

Here we discuss our work on both arithmetic and analytic aspects of the novel class of "signed" partition enumerations involving multiplicative $f : \mathbb{N} \to \{0, \pm 1\}$. Following this, we discuss a few problems toward developing more general results on some families of signed partition numbers.

Let $f : \mathbb{N} \to \{0, \pm 1\}$, and for $n \in \mathbb{N}$ and any partition $\pi = (a_1, a_2, \ldots, a_k) \in \Pi[n]$ let

$$f(\pi) := f(a_1)f(a_2)\cdots f(a_k). \tag{1.2}$$

With this we define the $(f-)$*signed partition numbers*

$$\mathfrak{p}(n, f) = \sum_{\pi \in \Pi[n]} f(\pi). \tag{1.3}$$

Definition (1.3) generalizes several classical partition-related quantities, e.g., with the constant function 1 one has $\mathfrak{p}(n) = \mathfrak{p}(n, 1)$, and with the indicator function $\mathbf{1}_A$ for $A \subset \mathbb{N}$, the quantities $\mathfrak{p}(n, \mathbf{1}_A)$ are the *A-restricted partition numbers*. Many examples and families of restricted partitions numbers are well studied (see, e.g., [1, 10]).

2 Arithmetic work; q-series and periodic vanishings

We say a sequence $(a_n)_{\mathbb{N}}$ *vanishes on an arithmetic progression* (or has a *periodic vanishing*) if one has $a_{km+t} = 0$ for some $0 \leq t < m$ and all $k \geq 0$. In such a case we may say that a_n "vanishes on all $n \equiv r \pmod m$". The past two decades have seen a boom of work on periodic vanishings in the coefficients of various q-series; we note [3, 4, 12, 14, 16] as only a small portion of the extant literature.

For odd prime p let $\chi(n) = \chi_p(n) = (\frac{n}{p})$ denote the Legendre symbol. In addition to $\chi(\pi)$, defined via (1.2), for any partition $\pi = (a_1, a_2, \ldots, a_k)$ of any positive n, we set

$$[-\chi](\pi) := (-1)^k \chi(\pi) = (-1)^k \chi(a_1)\chi(a_2)\cdots\chi(a_k).$$

The quantities $\mathfrak{p}(n,\chi)$ and $\mathfrak{p}(n,-\chi)$ are the *Legendre-signed partition numbers.* We note that $\mathfrak{p}(n,\chi)$ and $\mathfrak{p}(n,-\chi)$ may equivalently be defined via

$$\prod_{r=1}^{p-1}(\chi(r)q^r;q^p)_\infty^{-1} = \sum_{n=0}^{\infty}\mathfrak{p}(n,\chi)q^n, \quad \text{and} \quad \prod_{r=1}^{p-1}(-\chi(r)q^r;q^p)_\infty^{-1} = \sum_{n=0}^{\infty}\mathfrak{p}(n,-\chi)q^n,$$

where $(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n)$ is the q-Pochhammer symbol.

In [8], using a number of *q-series* identities and symbolic manipulations, we establish the following vanishing result.

Theorem 2.1. *One has*

$$\mathfrak{p}(n,+\chi_5) = 0 \qquad \textit{for } n \equiv 2 \pmod{10},$$
$$\mathfrak{p}(n,-\chi_5) = 0 \qquad \textit{for } n \equiv 6 \pmod{10}.$$

In [9], using the Hardy-Littlewood method (in particular following Vaughan's work on partitions into primes [17]), vanishing results for $\mathfrak{p}(n,\chi_p)$ akin to Theorem 2.1 are ruled out for a large proportion of primes.

Theorem 2.2. *For odd prime $p \neq 5$ with $p \not\equiv 1 \pmod 8$, one has $\mathfrak{p}(n,\chi_p) \to \infty$ as $n \to \infty$. Thus, for said p the sequences $(\mathfrak{p}(n,\chi_p))_{\mathbb{N}}$ do not vanishing identically on any arithmetic progression.*

In light of Theorem 2.2, we investigate if any $p \equiv 1 \pmod 8$ exhibit vanishings like those of Theorem 2.1. Thus far, numerical trials have produced only one further such prime, namely $p = 17$. The confirmation of periodic vanishing in the sequences $(\mathfrak{p}(n,\pm\chi_{17}))_{\mathbb{N}}$ is established in the author's *in preparation* manuscript [6].

Theorem 2.3. *One has*

$$\mathfrak{p}(n,\chi_{17}) = 0 \qquad \textit{for } n \equiv 17, 19, 25, 27 \pmod{34}.$$

Equivalently, $\mathfrak{p}(n,\chi_{17}) = 0$ whenever n is both odd and $1-24n$ is congruent to a quartic residue $\pmod{17}$. Further, one has

$$\mathfrak{p}(n,-\chi_{17}) = 0 \qquad \textit{for } n \equiv 11, 15, 29, 33 \pmod{34}.$$

Equivalently, $\mathfrak{p}(n,-\chi_{17}) = 0$ whenever n is both odd and $1-24n$ is congruent to a quadratic-nonquartic residue $\pmod{17}$.

There is compelling evidence (both empirical and theoretical) for the following.

Conjecture 2.4. The only odd primes for which $\mathfrak{p}(n,\chi_p)$ and $\mathfrak{p}(n,-\chi_p)$ vanish on some arithmetic progressions $\pmod{2p}$ are 5 and 17.

A "soft" explanation of the periodic vanishing demonstrated by $\mathfrak{p}(n,\chi_5)$ is provided by the following asymptotic formula.

Theorem 2.5 ([9, Thm. 1.7]). *As $n \to \infty$ one has*

$$\mathfrak{p}(n, \chi_5) = \mathfrak{a}_5 n^{-3/4} \exp\Big(\tfrac{1}{2}\kappa\sqrt{\tfrac{4}{5}n}\Big)\big[1 + (-1)^n \mathfrak{b}_5 + \mathfrak{d}_5 \cos\big(\tfrac{2\pi}{5}n - \tfrac{\pi}{10}\big) + O(n^{-1/5})\big], \quad (2.1)$$

where

$$\kappa = \pi\sqrt{\tfrac{2}{3}}, \quad \mathfrak{a}_5 = \left(\frac{3+\sqrt{5}}{960}\right)^{1/4}, \quad \mathfrak{b}_5 = \frac{3-\sqrt{5}}{2}, \quad \textit{and} \quad \mathfrak{d}_5 = \sqrt{2(5-\sqrt{5})}.$$

Ignoring the error term $O(n^{-1/5})$ in (2.1) and considering the 10-periodic term

$$\mathfrak{S}(n) := 1 + (-1)^n\left(\frac{3-\sqrt{5}}{2}\right) + \sqrt{2(5-\sqrt{5})}\,\cos\left(\frac{2\pi n}{5} - \frac{\pi}{10}\right),$$

it is surprising (and suggestive) to find that

$$\mathfrak{S}(2) = 0 \qquad \text{and} \qquad \mathfrak{S}(n) \neq 0 \quad \text{for } 1 \leq n \leq 10 \text{ with } n \neq 2.$$

3 Asymptotic results on some $\mathfrak{p}(n, f)$

When f assumes both positive and negative values one expects these signs to cause cancellations in the sums $\mathfrak{p}(n, f)$. We recall the Möbius μ and Liouville λ functions from prime number theory: If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ with distinct primes p_i and all $a_i \geq 1$, then

$$\lambda(n) := (-1)^{a_1+\cdots+a_r} \qquad \text{and} \qquad \mu(n) := \begin{cases} (-1)^r & \text{if all } a_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The following result is an immediate corollary of the main results of [7].

Theorem 3.1. *For all $\varepsilon > 0$, as $n \to \infty$ one has*

$$\mathfrak{p}(n, \mu) = O\big(e^{(1+\varepsilon)\sqrt{n}}\big) \qquad \textit{and} \qquad \mathfrak{p}(n, \lambda) = O\big(e^{(\frac{1}{2}+\varepsilon)\kappa\sqrt{n}}\big), \quad (3.1)$$

where $\kappa = \pi\sqrt{2/3}$. In addition, for positive integer k, as $k \to \infty$ one has

$$\log \mathfrak{p}(2k, \mu) \sim \sqrt{2k} \qquad \textit{and} \qquad \log \mathfrak{p}(2k, \lambda) \sim \tfrac{1}{2}\kappa\sqrt{2k}. \quad (3.2)$$

Given the relations of (3.2), it is natural to consider to what extent those relations "extend" to odd n. In [5] this question is answered under mild assumptions on the zeros of the Riemann zeta function $\zeta(s)$. Let $\Theta := \sup\{\mathrm{Re}(\rho) : \zeta(\rho) = 0\}$. It is well known that $\frac{1}{2} \leq \Theta \leq 1$; the assertion that $\Theta = \frac{1}{2}$ is the Riemann Hypothesis (RH).

Again for odd primes p let $\chi_p(n)$ denote the Legendre symbol $(\frac{n}{p})$. In [9], asymptotic formulae for different families of *Legendre-signed partition numbers* $\mathfrak{p}(n, \chi_p)$ are established, where primes are separated by their residue modulo 8. Of particular interest are the asymptotics for $\mathfrak{p}(n, \chi_p)$ for $p \equiv 3 \pmod 4$; indeed, in such cases the generating functions for the sequences $(\mathfrak{p}(n, \chi_p))_{\mathbb{N}}$ lack "modular" functional equations, as happens when $p \equiv 1 \pmod 4$. Because the exact first-order asymptotic formulae for $\mathfrak{p}(n, \chi_p)$ with $p \equiv 3 \pmod 4$ involve complicated constants, we present the following simpler result which is Corollary 1.6 in [9].

Theorem 3.2. *Let $\kappa := \pi\sqrt{2/3}$. If $p \equiv 7 \pmod 8$, then as $n \to \infty$ one has*

$$\mathfrak{p}(n, \chi_p) \asymp n^{\sqrt{p}L(1,\chi_p)/4\pi - 3/4} \exp\Big(\tfrac{1}{2}\kappa\sqrt{(1-\tfrac{1}{p})n}\Big),$$

where $L(s, \chi_p)$ is the Dirichlet L-function for χ_p. If $p \equiv 3 \pmod 8$, then as $n \to \infty$ one has the stronger relation

$$\mathfrak{p}(n, \chi_p) \sim \mathfrak{a}_p n^{\sqrt{p}L(1,\chi_p)/4\pi - 3/4} \exp\Big(\tfrac{1}{2}\kappa\sqrt{(1-\tfrac{1}{p})n}\Big),$$

where

$$\mathfrak{a}_p = \left(\frac{p-1}{384\,p^2}\right)^{\frac{1}{4}} \exp\left(\frac{\sqrt{p}L(1,\chi)}{2\pi}\left(\gamma + \frac{1}{2}\log\left(\frac{384}{p(p-1)}\right) - \frac{L'(1,\chi)}{L(1,\chi)}\right)\right),$$

and γ is the Euler-Mascheroni constant.

4 Proposed research

Here we discuss several directions of potential future research.

4.1 **Vanishings of Legendre-signed partition numbers.** Theorems 2.5 and 2.2 indicate that sequences $\mathfrak{p}(n, \chi_p)_{\mathbb{N}}$ involving primes $p \equiv 1 \pmod 8$ should be further explored. Empirical computations have been done, and these computations further indicate that the periodic vanishings of $\mathfrak{p}(n, \pm\chi_5)$ and $\mathfrak{p}(n, \pm\chi_{17})$ are indeed quite rare.

Problem 4.1. Establish the presence of periodic vanishings, or lack thereof, in sequences $(\mathfrak{p}(n, \pm\chi_p))_{\mathbb{N}}$ with $p \equiv 1 \pmod 8$. In particular, resolve Conjecture 2.4.

When $p \equiv 1 \pmod 4$, the generating function

$$\prod_{r=1}^{p-1} (\chi_p(r)q; q^p)_\infty^{-1} = 1 + \sum_{n=1}^{\infty} \mathfrak{p}(n, \chi_p) q^n$$

can be expressed as a quotient Dedekind's $\eta(z)$ function and Jacobi θ-functions, and thus enjoys a modular-like functional equation. This functional transformation relation allows one to give a convergent series representation for the coefficients $\mathfrak{p}(n, \chi_p)$, à-la Rademacher's convergent series for the ordinary partition numbers $\mathfrak{p}(n, 1)$.

Specifically, using the abbreviations $\kappa = \pi\sqrt{2/3}$ and $\lambda_n = \sqrt{n - 1/24}$, Rademacher establishes [15] that

$$\mathfrak{p}(n) = \kappa(384)^{-\frac{1}{4}} \lambda_n^{\frac{3}{4}} \sum_{k=1}^{\infty} A_k(n) k^{-1} I_{\frac{3}{2}}(\kappa\lambda_n/k), \tag{4.1}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind, and $A_k(n)$ is related to the classical Kloosterman sums.

For fixed $p \equiv 1 \pmod 4$ let

$$\mathfrak{L}_k(n) := \sum_{0<h\le k}{}' \exp\big\{\pi i \Lambda(h,k) - 2\pi i h n/k\big\}, \tag{4.2}$$

where $\Lambda(h, k)$ is a certain "character-twisted" Dedekind sum. Specifically,

$$\Lambda(h,k) = \tfrac{1}{2}\big\{s_\chi(h,k) - s_\chi(2h,k)\big\} + \tfrac{1}{2}\big\{s(2h,k) - s(2hp,k)\big\}, \tag{4.3}$$

$$s_\chi(h,k) := \sum_{\mu \bmod [k,p]} \chi(\mu)(\!(h\mu/k)\!)(\!(\mu/[k,p])\!), \tag{4.4}$$

where $[k,p] = \mathrm{lcm}(k,p)$, and $(\!(x)\!) = 0$ for $x \in \mathbb{Z}$ and $(\!(x)\!) = x - [x] - \frac{1}{2}$ otherwise. We note that when $(k,p) = 1$, our $s_\chi(h,k)$ agrees with the $s_\chi(h,k)$ in Berndt's notation [2].

Applying Rademacher's and Lehner's methods from, e.g., [15] and [13], we find that for $p < 24$, one has

$$\begin{aligned} \mathfrak{p}(n,\chi_p) = &\sum_{\substack{k=1 \\ 2\nmid k,\ p\nmid k}}^{\infty} \frac{1}{k}\left\{\lambda_k \mathfrak{L}_k(n) + \lambda_{2k}\mathfrak{L}_{2k}(n)\right\} I_1\big(f(k,n)\big), \\ &+ \sum_{\substack{k=1 \\ 4|k,\ p\nmid k}}^{\infty} \frac{1}{k}\left\{\lambda_k \mathfrak{L}_k(n)\right\} I_1\big(f(k,n)\big)/k + \sum_{\substack{k=1 \\ 2\nmid k,\ p|k}}^{\infty} \left\{\mathfrak{L}_k^+(n)\right\} I_1\big(g(k,n)\big)/k, \end{aligned} \tag{4.5}$$

where the λ_k are products of at most $2p$ fixed cosecant terms, I_1 is the modified Bessel function of the first kind, and f and g are elementary functions. The quantity $\mathfrak{L}_k^+(n)$ here is a modified version of the sum (4.2), wherein the sum in (4.2) is changed to only sum over those $h \pmod k$ such that $(h,k) = 1$ and $\chi(h) = +1$.

Determination of the sums $\mathfrak{L}_k$ requires detailed knowledge of $24k\Lambda(h,k) \pmod{48k}$. A number of lemmata on these congruences have been established by the author already; completion of a handful of further results will allow for Salié-like formulae for the Kloosterman sums

$$\mathfrak{L}_q(n,m) := \sum_{h \pmod q}{}' \exp\left\{\pi i \Lambda(h,q) - 2\pi i(hn + \overline{2h}m)/q\right\} \qquad (q = p^\alpha),$$

where $h\bar{h} \equiv 1 \pmod q$.

4.2 **Different families of signed partition numbers.** Recent and ongoing work with J. McLaughlin establishes further novel examples of periodic vanishings in the spirit of signed partitions, as seen in the following result.

Theorem 4.2 ([1])**.** *If one defines the sequence a_n via*

$$A(q) := \sum_{n=0}^{\infty} a_n q^n = \frac{(q,q^3,q^4,q^9,q^{10},q^{12};q^{13})_\infty}{(-q^2,-q^5,-q^6,-q^7,-q^8,-q^{11};q^{13})_\infty},$$

then one has

$$a_{13n+3} = a_{13n+9} = a_{13n+11} = 0 \qquad \text{for all } n \geq 0.$$

Proofs for vanishings modulo other select primes (for the corresponding q-series) are in progress, and empirical evidence suggests that these select primes with such vanishings are extremely rare.

Problem 4.3. Determine which primes $p \equiv 1 \pmod 4$ exhibit periodic vanishings in their signed partition numbers akin to those of Theorem 4.2.

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[1]In-preparation manuscript joint with J. McLaughlin

4.3 **More general theory.** The discussions of the previous sections have only involved arithmetic functions taking values in $\{0, \pm 1\}$. However, it is evident that the questions and methods discussed can be readily ported to examples involving more general weights, especially those involving characters χ with $|\chi| = 1$.

Problem 4.4. Examine sequences $\mathfrak{p}(n, \chi)$ with general Dirichlet characters χ, and find examples (if any exist) of characters χ which produce periodic vanishings in $\mathfrak{p}(n, \chi)$.

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